PRIKRY-RIZED MITCHELL, TREE PROPERTIES AND REFLECTION

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Abstract. We develop a new method for combining Mitchell forcing with Extender-Based Prikry forcing. As an application we produce a model of ZFC where $\mathsf{GCH}_{<\kappa}$, $\mathrm{Refl}(\kappa^+)$ and the ineffable tree property $\mathrm{ITP}(\kappa^{++})$ hold simultaneously.

1. Introduction

Suppose that all the "small" substructures of a given mathematical structure (a group, a graph, etc) witness a property φ. Does the entire structure satisfy φ? If this turns out to be the case one says that *compactness* holds. Roughly speaking, compactness is the phenomenon by which the local behavior of a mathematical structure determines its global nature.

As a fact of nature, compactness holds when "small" means finite. For instance, a theorem of De Bruijn and Erdös [BE51] establishes that a graph $\mathcal{G}$ has *chromatic number* $\leq n$ (for a fixed $n \in \mathbb{N}$) provided all of its finite subgraphs $\mathcal{H}$ have chromatic number $\leq n$. In mathematical logic and set theory two of the most prominent classical examples are Gödel's *Compactness Theorem* for first-order logic and the *Lévy reflection theorem*. Besides of these, compactness has found several applications in other areas or pure mathematics; such as Ramsey Theory, Algebra or Topology.

What if the small substructures are – rather than just finite – of size less than κ for a cardinal $\kappa \geq \aleph_1$? Should one still expect forms of compactness? In general, compactness fails at the level of $\aleph_1$; namely, when the substructures are countable. In contrast, compactness can hold for higher cardinals, but this typically requires the existence of *large cardinals*.

This paper is concerned with two prominent set-theoretic manifestations of the compactness phenomenon – the *tree property* and *stationary reflection*. Given an uncountable regular cardinal κ the *tree property* holds at κ (in symbols, $\mathrm{TP}(\kappa)$) if every κ-tree T has a cofinal branch. *Stationary reflection holds at* κ (in symbols, $\mathrm{Refl}(\kappa)$) if every stationary set $S \subseteq \kappa$ *reflects*; namely, there is $\alpha < \kappa$ of uncountable cofinality such that $S \cap \alpha$ is stationary in α.

The tree property at $\aleph_0$ is simply the conclusion of König's infinity lemma. In contrast, by Aronszajn, the tree property fails at the first uncountable

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cardinal, $\aleph_1$. Later, Mitchell proved in [Mit72] that $\mathrm{TP}(\aleph_2)$ is consistent (i.e., non contradictory with the ZFC axioms). He started with a type of large cardinal called *weakly compact* and used what is now called Mitchell forcing to turn it into $\aleph_2$. By Silver, this large cardinal hypothesis is necessary.

The tree property is closely tied with cardinal arithmetic. To get the tree property at $\aleph_2$, the *Continuum Hypothesis* (CH) must fail. More generally, Specker [Spe90] showed that the tree property at the double successor of a singular strong limit cardinal implies the failure of the *Singular Cardinal Hypothesis* (SCH). SCH is a parallel of CH for singular cardinals. The first result in this vein is due to Cummings and Foreman [CF98], who proved that $\mathrm{TP}(\kappa^{++})$ is consistent for κ singular strong limit.

The other property under consideration is stationary reflection. It is a classical result of Jensen that in Gödel's constructible universe $\mathrm{Refl}(\kappa)$ holds if and only if κ is weakly compact, thus affirming its large-cardinal strength. Regarding successors of singular cardinals, Magidor showed in [Mag82] that $\mathrm{Refl}(\aleph_{\omega+1})$ is consistent with ZFC. Recently, Hayut and Unger [HU20] reduced the large cardinal assumptions employed by Magidor.

An ongoing ambitious program in set theory is to determine the extent to which various compactness principles can coexist. The focus of this paper is combining stationary reflection at the successor of a singular κ with the tree property at the double successor of κ. In the course of this enterprise, this paper shall introduce a new technology combining Mitchell's classical forcing [Mit72] with Gitik–Magidor Extender–Based Prikry forcing [GM94].

A precursor of this technology traces back to Cummings–Foreman proof of the tree property at the double successor of a singular [CF98]. In that paper the authors interleaved Prikry forcing into the Mitchell poset with an eye kept at singularizing a cardinal while carrying out the tree property construction. Ever since Mitchell-like posets subsuming other Prikry-type forcings have been used in various similar constructions [Sin16, FHS18, Pov20].

However, none of these posets were *themselves* of Prikry-type, meaning there do not satisfy a Prikry lemma. In this paper we show that it is possible to combine Mitchell forcing with the Gitik-Magidor Extender-Based Prikry forcing into a poset that has the Prikry property in its own right. We believe our construction can be generalized to combine other Prikry type posets, and so get a variety of forcings.

Then we feed our Mitchell-Prikry poset into the iteration framework of [PRS22] to produce the following configuration:

Theorem. *Assume* GCH. *Suppose that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals with limit κ and λ is supercompact with $\lambda > \kappa$. Then there is a generic extension of the set-theoretic universe where:*

1. *κ is strong limit and $\lambda = \kappa^{++}$;*
2. *$\mathsf{GCH}_{<\kappa}$ holds but $2^\kappa = \lambda$;*
3. *$\mathrm{Refl}(\kappa^+)$ holds;*
4. *$\mathrm{TP}(\lambda)$ holds. Actually, the* Ineffable Tree Property (ITP) *holds.*

ITP is a strengthening of TP, isolated by Magidor [Mag74], and more recently studied by Weiß [Wei12]. Just like the tree property captures the combinatorial essence of weak compactness, ITP captures supercompactness – given an inaccessible κ, ITP(κ) holds if and only if κ is supercompact.

Let us now outline the content of the manuscript. In Section 2, we define the new Prikry-type forcing combining the Mitchell poset, $\mathbb{M}$, from [Mit72] with the Gitik-Magidor *Extender-Based Prikry forcing* [GM94]. This poset will be the initial step in the iteration leading to the above stated theorem. Section 3 presents a variant of Sharon's functor from [PRS22], designed to destroy non-reflecting stationary sets. In Section 4, we describe the Σ-Prikry-style iteration $\mathbb{P}_\lambda$, which will be used in our construction. In Section 5 we prove our theorem about stationary reflection and the tree property. The key technical result is that ITP(λ) holds after forcing with $\mathbb{P}_\lambda$. We build on previous works by Gitik [Git11] and Hatchman–Sinapova [HS19]. Our proof is considerably more complex than its ancestors [Git11, HS19] for $\mathbb{P}_\lambda$ is more involved than the forcings used in those works.

The paper is largely self-contained but some parts require acquaintance with the theory of Σ-Prikry forcings [PRS22, PRS23]. The reader will be provided with precise references when this is necessary. As in those papers, here we use the following notation: Given a projection $\pi : \mathbb{Q} \to \mathbb{P}$, and conditions $q, r \in Q$, we say that $q \leq^\pi r$ if $q \leq_{\mathbb{Q}} r$ and $\pi(q) = \pi(r)$. We shall denote by $\mathbb{Q}^\pi$ the poset $(Q, \leq^\pi)$.

2. A Mitchell-like Extender Based Forcing

In this section we introduce a hybrid between the Mitchell forcing from [Mit72] and the Gitik-Magidor Extender Based Prikry forcing from [GM94]. The motivation is to devise a poset that forces the tree property at the doule successor of a singular cardinal and has the nice properties of diagonal extender based forcing (particularly, when it comes to stationary reflection).

2.1. **Definition of $\mathbb{M}$.** Assume GCH. Through this section $\Sigma := \langle \kappa_n \mid n < \omega\rangle$ is an increasing sequence of $(\lambda+1)$-strong cardinals such that λ is a regular cardinal with $\lambda > \kappa := \sup(\Sigma)$. For each $n < \omega$ we fix a $(\kappa_n, \lambda+1)$-extender E_n witnessing that κ_n is $(\lambda+1)$-strong. Let $\mathbb{P}$ be the Gitik-Magidor *Extender-based Prikry forcing* (EBPF) defined with respect to the sequence of extenders $\vec{E}$ [Git10, §2]. The present section is self-contained but familiarity with the notations in [Git10, §2] is assumed. In what follows $\mathcal{R}$ will denote the regular cardinals in $[\kappa^+, \lambda)$.

We will need a couple of technical observations about the EBPF and its projections. Given $p = \langle f^p_0, \dots, f^p_{\ell-1}, (a^p_\ell, A^p_\ell, f^p_\ell), \dots\rangle \in \mathbb{P}$ and $\alpha \leq \lambda$ such that every member of $\langle a^p_n \cap \alpha \mid n \geq \ell\rangle$ contains a $\leq_{E_n}$-maximal element, one can define its putative restriction $p \restriction \alpha$ to $\mathbb{P} \restriction \alpha := \mathbb{P}_{\langle E_n \restriction \alpha \mid n<\omega\rangle}$ as follows:

$$p \restriction \alpha := \langle f^p_0 \restriction \alpha, \dots, f^p_{\ell-1} \restriction \alpha, (a^p_\ell \cap \alpha, \pi_{\mathrm{mc}(a^p_\ell), \mathrm{mc}(a^p_\ell \cap \alpha)}``A^p_\ell, f^p_\ell \restriction \alpha), \dots\rangle.$$

Lemma 2.1 ([Git10, Lemma 2.2])**.** *Let $\theta \in \mathcal{R}$ and $n < \omega$. Then, the partial order $\leq_{E_n} \restriction(\theta \times \theta)$ is κ_n-directed. Moreover, for every $a \in [\theta]^{<\kappa_n}$ there are θ-many $\alpha < \theta$ such that $\alpha \geq_{E_n} \beta$ for all $\beta \in a$.*

If $\alpha \in \mathcal{R}$ and $p \in \mathbb{P}$ then one can use Lemma 2.1 to produce a condition $q \in \mathbb{P}$, $q \leq^* p$, with the same Cohen functions (i.e., $f^q_n = f^p_n$ for all $n < \omega$) and such that $q \restriction \alpha \in \mathbb{P} \restriction \alpha$. Pushing this idea further it is possible to isolate a $\leq^*$-dense subposet $\mathbb{Q}$ of $\mathbb{P}$ for which $p \restriction \alpha$ is always well-defined. Specifically, let us consider $\mathbb{Q}$ the subposet of $\mathbb{P}$ whose universe is

$$\{p \in \mathbb{P} \mid \forall n \geq \ell(p)\, \forall \alpha \in \mathcal{R}\ (\mathrm{mc}(a^p_n \cap \alpha) \text{ exists})\}.$$

The following is a routine verification.

Lemma 2.2. *$\mathbb{Q}$ is $\leq^*$-dense in $\mathbb{P}$.* □

By virtue of Lemma 2.2, $\mathbb{Q}$ and $\mathbb{P}$ are forcing equivalent. As a result, we do not lose any generality by assuming that our EBPF poset is $\mathbb{Q}$. In a slight abuse of notation we shall keep denoting this poset by $\mathbb{P}$.

There is a natural strengthening of the $\leq^*$-ordering enjoying of better closure properties – this is the so-called *fusion ordering.*

Definition 2.3 (Fusion ordering)**.** Let $p, q \in \mathbb{P}$ be conditions with length ℓ and $k \geq \ell$. We shall write $p \leq^{*,k} q$ if $p \leq^* q$ and the following clauses hold:

(1) $a^p_n = a^q_n$ for $n \in [\ell, k]$;
(2) $A^p_n = A^q_n$ for $n \in [\ell, k]$;

We shall write $p \leq^{*,k,-} q$ if $p \leq^* q$ and Clause (1) above hold.

Lemma 2.4. *For each $k < \omega$, $\langle \mathbb{P}, \leq^{*,k} \rangle$ is κ_{k+1}-closed.* □

The orderings $\leq^{*,k,-}$ and $\leq^{*,k}$ will play a key role in our discussions about the tree property. On a similar vein, we will also need to identify various natural projections associated to the EBPF poset:

Lemma 2.5. *Fix $\beta < \alpha$ in $\mathcal{R}$. Then,*

$$\cdot \restriction \beta \colon p \mapsto p \restriction \beta,$$

is a length-preserving projection from $\mathbb{P} \restriction \alpha$ to $\mathbb{P} \restriction \beta$.

These projections commute and, for each $p \in \mathbb{P}$ and $\vec{\nu} \in \prod_{\ell(p) \leq i \leq k} A^p_i$

$$(p^\frown \vec{\nu}) \restriction \alpha = (p \restriction \alpha)^\frown \langle \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}(\nu_i) \mid \ell(p) \leq i \leq k \rangle,$$

where $p^\frown \vec{\nu}$ is the weakest extension of p using $\vec{\nu}$. □

With the above results at hand we can define the *Mitchell-like Extender Based Prikry forcing* (MEBPF) $\mathbb{M}$ as follows:

Definition 2.6. A condition in $\mathbb{M}$ is a pair (p, c) where $p \in \mathbb{P}$, c is a function with $\mathrm{dom}(c) \in [\mathcal{R}]^{\leq \kappa}$ and for each $\alpha \in \mathrm{dom}(c)$ the following hold:

(1) $c(\alpha) \colon [\prod_{n \geq \ell(p)} \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}\text{“}A^p_n]^{<\omega} \to V^{\mathbb{P} \restriction \alpha}$;[1]

[1] More verbosely, $\mathrm{dom}(c(\alpha))$ consists of finite sequences $\vec{\pi} = \langle \pi_{\ell(p)}, \ldots, \pi_k \rangle$ in the product of the measure one sets of p and its outcome $c(\alpha, \vec{\pi})$ is a $\mathbb{P} \restriction \alpha$-name.

(2) $\mathbb{1} \Vdash_{\mathbb{P}\restriction\alpha}$ "$c(\alpha, \vec{\pi}) \in \dot{\mathrm{Add}}(\kappa^+, 1)$", for all $\vec{\pi} \in \mathrm{dom}(c(\alpha))$;
(3) $(p \restriction \alpha)^\frown \vec{\sigma} \Vdash_{\mathbb{P}\restriction\alpha} c(\alpha, \vec{\sigma}) \leq c(\alpha, \vec{\pi})$, for all $\vec{\pi} \sqsubseteq \vec{\sigma}$ in $\mathrm{dom}(c(\alpha))$.

Given $(p, c), (q, d) \in \mathbb{M}$ we write $(p, c) \leq^* (q, d)$ if and only if:

(I) $p \leq^* q$;
(II) $\mathrm{dom}(c) \supseteq \mathrm{dom}(d)$;
(III) for $\alpha \in \mathrm{dom}(d)$ and $\vec{\pi} = \langle \pi_\ell, \ldots, \pi_k \rangle \in \prod_{\ell \leq i \leq k} \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}$ "A^p_i,

$$(p \restriction \alpha)^\frown \vec{\pi} \Vdash_{\mathbb{P}\restriction\alpha} \text{“}c(\alpha, \vec{\pi}) \leq d(\alpha, \langle \pi_{\mathrm{mc}(a^p_i \cap \alpha), \mathrm{mc}(a^q_i \cap \alpha)}(\pi_i) \mid \ell \leq i \leq k \rangle)\text{”}.$$

Remark 2.7. It is routine to check that clause (III) above is well-defined. This is a consequence of $p \leq^* q$ and the commutativity of the projections associated to the EBPF. The idea behind $\mathbb{M}$ is the following. As usual, the EBPF-*part* (incarnated by p) contributes to generating an initial segment $\vec{\nu} := \langle \nu_{\ell(p)}, \ldots \nu_k \rangle$ of one of the eventual EBPF-generics. This $\vec{\nu}$ guides the *Cohen part* (represented by c) to collapse all V-regular cardinals in (κ^+, λ).

Let us now define the n-point extensions:

Definition 2.8. Let $(p, c) \in \mathbb{M}$ and $\nu \in A^p_\ell$. Denote $(p, c)^\frown \nu := (p^\frown \nu, c^\frown \nu)$ where $c^\frown \nu$ is the function defined as follows:

(1) $\mathrm{dom}(c^\frown \nu) := \mathrm{dom}(c)$;
(2) $\mathrm{dom}((c^\frown \nu)(\alpha))$ is the collection of all $\langle \pi_{\ell+1}, \ldots, \pi_k \rangle$ such that

$$\langle \pi_{\mathrm{mc}(a^p_\ell), \mathrm{mc}(a^p_\ell \cap \alpha)}(\nu) \rangle^\frown \langle \pi_{\ell+1}, \ldots, \pi_k \rangle \in \mathrm{dom}(c(\alpha));$$

(3) for each $\vec{\pi} = \langle \pi_{\ell+1}, \ldots, \pi_k \rangle \in \mathrm{dom}((c^\frown \nu)(\alpha))$,

$$(c^\frown \nu)(\alpha, \vec{\pi}) := c(\alpha, \langle \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}(\nu) \rangle^\frown \langle \pi_{\ell+1}, \ldots, \pi_k \rangle).$$

In general, one defines $(p, c)^\frown \vec{\nu}$ by recursion on the length of $\vec{\nu}$. More explicitly, $(p, c)^\frown \vec{\nu} := (p^\frown \vec{\nu}, c^\frown \vec{\nu})$ where for each $\langle \pi_{|\vec{\nu}|+1}, \ldots, \pi_k \rangle$,

$$(c^\frown \vec{\nu})(\alpha, \vec{\pi}) := c(\alpha, \langle \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}(\nu_i) \mid \ell \leq i \leq |\vec{\nu}| \rangle^\frown \langle \pi_{|\vec{\nu}|+1}, \ldots, \pi_k \rangle).$$

Remark 2.9. Roughly speaking, $c^\frown \nu$ is the *shift of* c by ν. Namely, $c^\frown \nu$ is the natural restriction of c where the ℓ-entry is fixed to be the projections of the Prikry point $\vec{\nu}$. In general, $c^\frown \vec{\nu}$ is the natural restriction of c where the first $|\vec{\nu}|$-many coordinates are fixed to be projections of $\vec{\nu}$. Using Lemma 2.5 and the definition of $\mathbb{M}$ it is routine to check that $(p, c)^\frown \vec{\nu}$ is a condition.

Finally, let us define the order $\leq$ of $\mathbb{M}$:

Definition 2.10. Given $(p, c), (q, d) \in \mathbb{M}$ let us write $(p, c) \leq (q, d)$ if there is a sequence $\vec{\nu} \in \prod_{\ell(q) \leq i \leq \ell(p)} A^q_i$ such that $(p, c) \leq^* (q, d)^\frown \vec{\nu}$.

Next, we prove the main properties of $\mathbb{M}$. We will show that $\mathbb{M}$ has the *Complete Prikry Property* (see [PRS22, §2]), analyze the cardinal structure, and finally prove that $\mathbb{M}$ projects to $\mathbb{M} \restriction \alpha$.[2]

[2]In analogy with $\mathbb{P}\restriction\alpha$, $\mathbb{M}\restriction\alpha$ denotes the poset defined as $\mathbb{M}$ but using $\mathbb{P}\restriction\alpha$ as EBPF-part members and Cohen-parts c restricted to ordinals in $\mathcal{R} \cap \alpha$.

2.2. **Prikry property.** For the next lemma, recall the definition of property $\mathcal{D}$ from [PRS22, Definition 2.12].

Lemma 2.11. *$\mathbb{M}$ has property $\mathcal{D}$.*

Proof. Let $a = (p, c) \in \mathbb{M}$, $n < \omega$ and $\vec{r} = \langle \vec{\nu}_\alpha \mid \alpha < \kappa_{\ell(p)+n-1} \rangle$ be an injective enumeration of the product of the first n-many measure one sets of p. Our aim is to show that **I** has a winning strategy in the diagonalizability game $\eth_{\mathbb{M}}(a, \vec{r})$. For simplicity, let us provide details just when $n = 1$. Write $\ell := \ell(p)$. At the beginning, **I** plays $a_0 := a$ and, in response, **II** plays some $(q_0, d_0) := b_0 \leq^* a_0{}^\frown \vec{\nu}_0$. In the next round, **I** plays a pair $a_1 := (p_1, c_1) \leq^* a_0$ attempting to *diagonalize* b_0. More explicitly,

$$p_1 := \langle f_0^{q_0}, \dots, f_{\ell-1}^{q_0}, (a_\ell^p, A_\ell^p, f_\ell^{q_0} \setminus a_\ell^p)), (a_{\ell+1}^{q_0}, A_{\ell+1}^{q_0}, f_{\ell+1}^{q_0}), \dots \rangle,$$

and c_1 is a function with

- $\mathrm{dom}(c_1) := \mathrm{dom}(d_0)$;
- $\mathrm{dom}(c_1(\alpha)) := [\prod_{i \geq \ell} \pi_{\mathrm{mc}(a_i^{p_1}), \mathrm{mc}(a_i^{p_1} \cap \alpha))} \text{“} A_i^{p_1}]^{<\omega}$,
- Fix $\vec{\pi} = \langle \pi_\ell, \cdots, \pi_k \rangle \in \mathrm{dom}(c_1(\alpha))$.

If $|\vec{\pi}| \geq 2$ and $\pi_\ell = \pi_{\mathrm{mc}(a_\ell^p), \mathrm{mc}(a_\ell^p \cap \alpha)}(\nu_0)$, set:

$$c_1(\alpha, \vec{\pi}) := d_0(\alpha, \langle \pi_{\ell+1}, \cdots, \pi_k \rangle).$$

Otherwise,

$$c_1(\alpha, \vec{\pi}) := \begin{cases} c_0(\alpha, \langle \pi_{\mathrm{mc}(a_i^{p_1} \cap \alpha), \mathrm{mc}(a_i^p \cap \alpha))}(\pi_i) \mid i \in [\ell, k] \rangle), & \text{if } \alpha \in \mathrm{dom}(c_0); \\ \{(\check{\varnothing}, \mathbb{1}_{\mathbb{P} \restriction \alpha})\}, & \text{otherwise.} \end{cases}$$

Claim 2.11.1. *$(p_1, c_1) \in \mathbb{M}$, $(p_1, c_1) \leq^* (p_0, c_0)$ and $(p_1, c_1)^\frown \nu_0 = b_0$.*

Proof of claim. Clearly, $(p_1, c_1)^\frown \nu_0 = b_0$. To see that $(p_1, c_1) \in \mathbb{M}$, it suffices to check that c_1 is well-defined and increasing when more Prikry points are chosen (see (2) and (3) of Definition 2.6). Clause (2) is fairly easy to verify. As for Clause (3) we argue as follows: Fix $\vec{\pi} \sqsubset \vec{\sigma}$ in $\mathrm{dom}(c_1(\alpha))$.

Case $\alpha \in \mathrm{dom}(d_0) \setminus \mathrm{dom}(c_0)$: If $\vec{\pi}$ witnesses the property of the first case then so does $\vec{\sigma}$. In that scenario, $(p_1 \restriction \alpha)^\frown \vec{\pi} = (q_0 \restriction \alpha)^\frown \langle \pi_{\ell+1}, \dots, \pi_k \rangle$ (and similarly for $\vec{\sigma}$). Thus, $(p_1 \restriction \alpha)^\frown \vec{\sigma} \Vdash_{\mathbb{P} \restriction \alpha}$ “$d_0(\alpha, \vec{\sigma} \setminus \ell) \leq d_0(\alpha, \vec{\pi} \setminus \ell)$” because (q_0, d_0) is a condition. In case $|\vec{\pi}| = 1$ but $\pi_\ell = \pi_{\mathrm{mc}(a_\ell^p), \mathrm{mc}(a_\ell^p \cap \alpha)}(\nu_0)$ it follows that $(p_1 \restriction \alpha)^\frown \vec{\sigma} = (q_0 \restriction \alpha)^\frown \langle \sigma_{\ell+1}, \dots, \sigma_k \rangle$ and clearly this forces $d_0(\alpha, \langle \sigma_{\ell+1}, \dots, \sigma_k \rangle) \leq \{\langle \check{\varnothing}, \mathbb{1}_{\mathbb{P} \restriction \alpha} \rangle\}$. In other case we fall into the second alternative and the desired property holds.

Case $\alpha \in \mathrm{dom}(c_0)$: If $\vec{\pi}$ is as in the first alternative one can argue as before that $(p_1 \restriction \alpha)^\frown \sigma$ forces $c_1(\alpha, \vec{\sigma}) \leq c_1(\alpha, \vec{\pi})$. Also, if $\pi_\ell \neq \pi_{\mathrm{mc}(a_\ell^p), \mathrm{mc}(a_\ell^p \cap \alpha)}(\nu_0)$ then both $\vec{\pi}$ and $\vec{\sigma}$ fall into the second alternative and one can use that $(p_1 \restriction \alpha)^\frown \vec{\sigma} \leq (p_0 \restriction \alpha)^\frown \langle \pi_{\mathrm{mc}(a_i^{p_1} \cap \alpha), \mathrm{mc}(a_i^p \cap \alpha)}(\sigma_i) \mid \ell \leq i \leq k \rangle$ and that $(p_0, c_0) \in \mathbb{M}$ to infer the desired property. So, we are left with the case where $|\vec{\pi}| = 1$ and $\pi_\ell = \pi_{\mathrm{mc}(a_\ell^p), \mathrm{mc}(a_\ell^p \cap \alpha)}(\nu_0)$. In that scenario

$$c_1(\alpha, \pi_\ell) = c_0(\alpha, \pi_\ell) \text{ and } c_1(\alpha, \vec{\sigma}) = d_0(\alpha, \langle \sigma_{\ell+1}, \dots, \sigma_{k+1} \rangle).$$

The proof will be completed once we check that

$$(\star) \qquad (p_1 \restriction \alpha)^\frown \vec{\sigma} \Vdash_{\mathbb{P}\restriction\alpha} d_0(\alpha, \langle \sigma_{\ell+1}, \dots, \sigma_k\rangle) \le c_0(\alpha, \pi_\ell).$$

Recall that $(q_0, d_0) \le^* (p_0, c_0)^\frown \nu_0$, which by definition implies

$$(q_0{\restriction}\alpha)^\frown(\vec{\sigma}\setminus\ell) \Vdash_{\mathbb{P}\restriction\alpha} d_0(\alpha, \vec{\sigma}\setminus\ell) \le c_0(\alpha, \langle\pi_\ell\rangle^\frown\langle \pi_{\mathrm{mc}(a_i^{q_0}\cap\alpha),\mathrm{mc}(a_i^{p}\cap\alpha)}(\sigma_i) \mid i \ge \ell{+}1\rangle).$$ [3]

Since $(q_0 \restriction \alpha)^\frown(\vec{\sigma} \setminus \ell) \le (p_0 \restriction \alpha)^\frown\langle \pi_{\mathrm{mc}(a_i^{p_1}\cap\alpha),\mathrm{mc}(a_i^{p}\cap\alpha)}(\sigma_i) \mid i \ge \ell + 1\rangle$,

$$(q_0{\restriction}\alpha)^\frown(\vec{\sigma}\setminus\ell) \Vdash_{\mathbb{P}\restriction\alpha} c_0(\alpha, \langle\pi_\ell\rangle^\frown\langle \pi_{\mathrm{mc}(a_i^{q_0}\cap\alpha),\mathrm{mc}(a_i^{p}\cap\alpha)}(\sigma_i) \mid i \ge \ell{+}1\rangle) \le c_0(\alpha, \pi_\ell)$$

because (p_0, c_0) was a condition. All in all, as $(p_1 \restriction \alpha)^\frown\vec{\sigma} = (q_0 \restriction \alpha)^\frown(\vec{\sigma} \setminus \ell)$, the last two expressions together imply that $(\star)$ holds. The above arguments also show that $(p_1, c_1) \le^* (p_0, c_0)$. □

After defining $a_1 := (p_1, c_1)$, **II** plays in response $b_1 \le^* (p_1, c_1)^\frown\nu_1$. In general, suppose that $\langle (a_\xi, b_\xi) \mid \xi < \zeta\rangle$, $\zeta \le \kappa_l$, has been formed according to the rules of $\Game_{\mathbb{M}}(a, \vec{r})$. If $\zeta = \xi + 1$ then let **I** play $a_\zeta = (p_\zeta, c_\zeta)$ where

$$p_\zeta := \langle f_0^{q_\xi}, \dots, f_{\ell-1}^{q_\xi}, (a_\ell^p, A_\ell^p, f_\ell^{q_\xi} \setminus a_\ell^p)), (a_{\ell+1}^{q_\xi}, A_{\ell+1}^{q_\xi}, f_{\ell+1}^{q_\xi}), \dots\rangle,$$

and c_ζ is defined as c_1 but using (a_ξ, b_ξ) instead of (a_0, b_0). Arguing as in the previous claim one shows that $a_\zeta := (p_\zeta, c_\zeta) \in \mathbb{M}$ and that $a_\zeta{}^\frown\nu_\xi = b_\xi$.

Otherwise,

$$p_\zeta := \langle \textstyle\bigcup_{\xi<\zeta} f_0^{q_\xi}, \dots, \bigcup_{\xi<\zeta} f_{\ell-1}^{q_\xi}, (a_\ell^p, A_\ell^p, (\bigcup_{\xi<\zeta} f_\ell^{q_\xi}) \setminus a_\ell^p)), (a_{\ell+1}^{p_\zeta}, A_{\ell+1}^{p_\zeta}, f_{\ell+1}^{p_\zeta}), \dots\rangle,$$

where the $(a_{\ell+1}^{p_\zeta}, A_{\ell+1}^{p_\zeta}, f_{\ell+1}^{p_\zeta})$'s are the result of taking $\le^*$-lower bounds on the sequence $\langle q_\xi \setminus \ell \mid \zeta < \xi\rangle$.[4] Next, define c_ζ as the function with:

- $\mathrm{dom}(c_\zeta) := \bigcup_{\xi<\zeta} \mathrm{dom}(c_\xi)$,
- $\mathrm{dom}(c_\zeta(\alpha)) := [\prod_{i\ge\ell} \pi_{\mathrm{mc}(a_i^{p_\zeta}),\mathrm{mc}(a_i^{p_\zeta}\cap\alpha)}$ “$A_i^{p_\zeta}]^{<\omega}$,

Let $\alpha \in \mathrm{dom}(c_\xi)$ and denote by Λ_α the collection of all $\xi < \zeta$ such that $\alpha \in \mathrm{dom}(c_\xi)$. Let $\vec{\pi} = \langle \pi_\ell, \dots, \pi_k\rangle$ in $\mathrm{dom}(c_\zeta(\alpha))$. To ease notations put

$$\vec{\pi}_{\xi,\alpha} := \langle \pi_{\mathrm{mc}(a_i^{p_\zeta}\cap\alpha),\mathrm{mc}(a_i^{p_\xi}\cap\alpha)}(\pi_i) \mid \ell \le i \le k\rangle.$$

For each $\xi < \eta$ in Λ_α we have, by construction,

$$(p_\zeta \restriction \alpha)^\frown\vec{\pi} \le^* (p_\eta \restriction \alpha)^\frown\vec{\pi}_{\eta,\alpha} \le^* (p_\xi \restriction \alpha)^\frown\vec{\pi}_{\xi,\alpha}$$

and

$$(p_\eta \restriction \alpha)^\frown\vec{\pi}_{\eta,\alpha} \Vdash_{\mathbb{P}\restriction\alpha} \text{“}\langle c_\xi(\alpha, \vec{\pi}_{\xi,\alpha}) \mid \xi \in \Lambda_\alpha \cap \eta\rangle \text{ is } \le_{\dot{\mathrm{Add}}(\kappa^+,1)}\text{-decreasing”}.$$

So, let $c_\zeta(\alpha, \vec{\pi})$ is a $\mathbb{P} \restriction \alpha$-name for

$$\bigwedge\{c_\xi(\alpha, \vec{\pi}_{\xi,\alpha}) \mid \xi \in \Lambda_\alpha\};$$

namely, a lower bound for the displayed sequence, as forced by $(p_\zeta \restriction \alpha)^\frown\vec{\pi}$.

[3]Note that here we used $\pi_{\mathrm{mc}(a_\ell^p),\mathrm{mc}(a_\ell^p\cap\alpha)}(\nu_0) = \pi_\ell$.

[4]Such lower bounds exists because $\zeta < \kappa_{\ell+1}$.

Then, we have that

$$(p_\zeta \restriction \alpha)^\frown \vec{\pi} \Vdash_{\mathbb{P}\restriction\alpha} \forall \xi \in \Lambda_\alpha \, (c_\zeta(\alpha, \vec{\pi}) \leq_{\dot{\mathrm{Add}}(\kappa^+,1)} c_\xi(\alpha, \vec{\pi}_{\xi,\alpha})),$$

By tweaking $c_\zeta(\alpha, \vec{\pi})$ a bit we can moreover assume that it is a $\mathbb{P}\restriction\alpha$-name for a member of $\dot{\mathrm{Add}}(\kappa^+, 1)$ as forced by the trivial condition. In a similar spirit, given $\vec{\pi} \sqsubseteq \vec{\sigma}$ in $\mathrm{dom}(c_\zeta(\alpha))$, one can argue that

$$(p_\zeta \restriction \alpha)^\frown \vec{\sigma} \Vdash_{\mathbb{P}\restriction\alpha} c_\zeta(\alpha, \vec{\sigma}) \leq c_\xi(\alpha, \vec{\pi}_{\xi,\alpha}).$$

Claim 2.11.2. *$(p_\zeta, c_\zeta) \in \mathbb{M}$, $(p_\zeta, c_\zeta) \leq^* (p_\xi, c_\xi)$ and $(p_\zeta, c_\zeta)^\frown \nu_\xi \leq^* b_\xi$.*

Proof of claim. The first claim is obvious attending to our previous considerations. The claim that $(p_\zeta, c_\zeta)^\frown \nu_\xi \leq^* b_\xi$ follows from our inductive assumption; in effect, $(p_\zeta, c_\zeta)^\frown \nu_\xi \leq^* (p_{\xi+1}, c_{\xi+1})^\frown \nu_\xi \leq^* b_\xi$. □

The above defines the ζth-move of **I**, $a_\zeta := (p_\zeta, c_\zeta)$ for $\zeta \leq \kappa_l$. Letting **II** play some $b_\zeta \leq^* {a_\zeta}^\frown \nu_\zeta$ for $\zeta < \kappa_l$ yields a sequence $\langle (a_\zeta, b_\zeta) \mid \zeta < \kappa_\ell \rangle$.

To show that $\langle b_\zeta \mid \zeta < \kappa_\ell \rangle$ is diagonalizable, let $a' := (p_{\kappa_\ell}, c_{\kappa_\ell})$. Then $a' \leq^* a$ and ${a'}^\frown \nu_\zeta \leq^* {a_{\zeta+1}}^\frown \nu_\zeta = b_\zeta$ for all $\zeta < \kappa_\ell$. This shows that **I** has a winning strategy in $\Game_{\mathbb{M}}(a, \vec{r})$, as wished. □

Lemma 2.12. *$\mathbb{M}$ has the Complete Prikry Property.*

Proof. Fix $a = (p, c) \in \mathbb{M}$, $n < \omega$ and U a $\leq^*$-open subset of $\mathbb{M}$, i.e. if $b \in U$ and $b' \leq^* b$, then $b' \in U$. We will find $a' \leq^* a$ with the following property: either every $b \leq a$ with length $\ell(a) + n$ is in U or all of them avoid U. For simplicity, assume $\ell(a) = 0$. The general case is analogous.

Claim 2.12.1. *There is $a_0 \leq^* a$ such that, for every $b \leq a_0$ with $\ell(b) = n$, if $b \in U$ then ${a_0}^\frown \vec{\nu} \in U$, where $\vec{\nu}$ is the unique such that $b \leq^* {a_0}^\frown \vec{\nu}$.*

Proof of claim. Let $\vec{r} = \langle \vec{\nu}_\xi \mid \xi < \kappa_n \rangle$ be an injective enumeration of $\prod_{i \leq n} A^p_i$ (i.e., of the n-point extensions of a). Let us play the game $\Game_{\mathbb{M}}(a, \vec{r})$ but this time restricting the moves of **II** to U when this choice is possible. Specifically, let us define by induction a sequence $\langle (a_\xi, b_\xi) \mid \xi < \kappa_n \rangle \subseteq \mathbb{M}$ with $a_\xi \leq^* a$ and $b_\xi \leq^* {a_\xi}^\frown \vec{\nu}_\xi$ and if there is $b \leq^* {a_\xi}^\frown \vec{\nu}_\xi$ in U then **II** picks such $b_\xi \in U$; otherwise, **II** chooses $b_\xi := {a_\xi}^\frown \vec{\nu}_\xi$. By Lemma 2.11 there is $a_0 \leq^* a$ diagonalizing $\langle b_\xi \mid \xi < \kappa_n \rangle$.

We claim that a_0 is as desired.[5] Let $b \leq a_0$ in U with $\ell(b) = n$. By definition, there is $\vec{\nu}_\xi$ such that $b \leq^* {a_0}^\frown \vec{\nu}_\xi$. Since a_0 diagonalizes the b_ξ's we have that $b \leq^* {a_\xi}^\frown \vec{\nu}_\xi$. In particular, b_ξ was chosen to be in U. Since ${a_0}^\frown \vec{\nu}_\xi \leq^* b_\xi \in U$ it follows (by $\leq^*$-openness) that ${a_0}^\frown \vec{\nu}_\xi \in U$, as well. □

Let us now move to the *Röwbottom part* of the argument for the CPP. Namely, we define $b \leq^* a$ such that $b = (q, d)$ and

$$\text{“}\forall \vec{\nu} \in \textstyle\prod_{i<n} A^q_i \; (b^\frown \vec{\nu} \in U)\text{” or “}\forall \vec{\nu} \in \prod_{i<n} A^q_i \; (b^\frown \vec{\nu} \notin U)\text{”}.$$

We give details for when $n = 2$ – this suffices to understand the general case.

[5] It is important for the forthcoming argument that the first n-many measure one sets appearing in a_0 are exactly $\prod_{i \leq n} A^p_i$. This is exactly what the proof of Lemma 2.11 shows.

For each $\nu \in A^a_0$ define

$$B^0_\nu := \{\eta \in A^{p_0}_1 \mid a^\frown\langle \eta, \nu\rangle \in U\} \text{ and } B^1_\nu := A^{p_0}_1 \setminus B^0_\nu.$$

For each $\nu \in A^a_0$ let B_ν be the unique of the above two sets which is $E_{\mathrm{mc}(a^{p_0}_1)}$-large. Put $B^0 := \{\nu \in A^a_0 \mid B_\nu = B^0_\nu\}$ and $B^1 := \{\nu \in A^a_0 \mid B_\nu = B^1_\nu\}$.

Let A_0 be the unique of the above which is $E_{\mathrm{mc}(a^{p_0}_0)}$-large and define $A_1 := \bigcap_{\nu \in A^{a_2}_0} B_\nu$, which is $E_{\mathrm{mc}(a^{p_0}_1)}$-large by completeness.

Define q as p but replacing the first two measure one sets by A_0 and A_1, respectively. Define d with the same domain and values as c_0 but the $\vec{\pi} \in \mathrm{dom}(d(\alpha))$'s are from the αth-projection of the measure one sets of q. $\square$

Combining the CPP of $\mathbb{M}$ (Lemma 2.12) with the following easy lemma we infer that $\mathbb{M}$ has both the *Strong Prikry Property* and the *Prikry Property*.

Lemma 2.13. *For each $n < \omega$, the poset*

$$\mathbb{M}_n := \{(p, c) \in \mathbb{M} \mid \ell(p) = n\}$$

is κ_n-directed-closed.

Proof. Let $D \subseteq \mathbb{M}_n$ be a directed subset of conditions with $|D| < \kappa_n$. Let q be a $\leq^*$-lower bound for the EBPF-part of every $(p, c) \in D$. Let c^* be the function defined similarly to c_ζ in page 7. Namely, the domain of c^* is $\bigcup_{(p,c)\in D} \mathrm{dom}(c)$ and for each $\alpha, \vec{\pi}$, $c^*(\alpha, \vec{\pi})$ is obtained by taking lower bounds of relevant $c(\alpha, \vec{\sigma})$, where $(p, c) \in D$. As argued in Claim 2.11.2, $(q, c^*) \in \mathbb{M}$. Clearly it defines a $\leq^*$-lower bound for the conditions in D. $\square$

Corollary 2.14.

(1) $\mathbb{M}$ has the Prikry Property; namely, for every sentence φ in the forcing language of $\mathbb{M}$ and every $a \in \mathbb{M}$ there is $b \leq^* a$ deciding φ.
(2) $\mathbb{M}$ has the Strong Prikry Property; namely, for every $a \in \mathbb{M}$ and $D \subseteq \mathbb{M}$ dense open there is $n < \omega$ and $b \leq^* a$ such that every $c \leq b$ with $\ell(c) \geq \ell(b) + n$ is in D.

2.3. **Cardinal structure.** Let us now analyze the cardinal structure of $V^{\mathbb{M}}$.

Lemma 2.15. *Forcing with $\mathbb{M}$ preserves all cardinals $\leq \kappa^+$.*

Proof. The preservation of cardinals $\leq \kappa$ follows from Corollary 2.14(1) and Lemma 2.13. Similarly, the prerservation of κ^+ can be established using Corollary 2.14(2) and the fact that $|\{(p, c)^\frown\vec{\nu} \mid \vec{\nu} \in [\prod_{n\geq \ell(p)} A^p_n]^{<\omega}\}| \leq \kappa$ for all $(p, c) \in \mathbb{M}$. The argument in both cases is standard, but further details can be found in [PRS19, Lemma 2.10]. $\square$

Lemma 2.16. *Forcing with $\mathbb{M}$ collapses all V-regular cardinals in (κ^+, λ).*

Proof. Let $\alpha \in (\kappa^+, \lambda)$ be a V-regular cardinal and G a $\mathbb{M}$-generic filter over V. Since $\mathbb{M}$ projects to $\mathbb{P}$ and this latter to $\mathbb{P}\restriction\alpha$ we can respectively derive $\mathbb{P}$

and $\mathbb{P}\restriction\alpha$-generics $\bar{G}$ and $\bar{G}_\alpha$. Working in $V[G]$ the putative αth-collapsing function is defined as follows:

$$c_\alpha := \bigcup\{c(\alpha,\vec{\nu}\restriction\alpha)_{\bar{G}_\alpha} \mid (p,c)\in G,\ \alpha\in\mathrm{dom}(c),\ p^\frown\vec{\nu}\in\bar{G}\},^6$$

where, as before, $\vec{\nu}\restriction\alpha := \langle \pi_{\mathrm{mc}(a_i^p),\mathrm{mc}(a_i^p\cap\alpha)}(\nu_i) \mid \ell(p)\leq i\leq|\vec{\nu}|\rangle$.

Claim 2.16.1. *c_α is well-defined.*

Proof. Let $c(\alpha,\vec{\nu}\restriction\alpha)_{\bar{G}_\alpha}$ and $d(\alpha,\vec{\eta}\restriction\alpha)_{\bar{G}_\alpha}$ be as above. These come with conditions $(p,c),(q,d)\in G$ with $\alpha\in\mathrm{dom}(c)\cap\mathrm{dom}(d)$.[7] Let $(r,e)\leq(p,c),(q,d)$ in G with $\ell(r)\geq\max\{\ell(p^\frown\vec{\nu}),\ell(q^\frown\vec{\eta})\}$. By definition, there are $\vec{\sigma},\vec{\rho}$ such that $(r,e)\leq^*(p,c)^\frown\vec{\sigma},(q,d)^\frown\vec{\rho}$. Clearly $\vec{\nu}\sqsubseteq\vec{\sigma}$ and $\vec{\eta}\sqsubseteq\vec{\rho}$.

Since $\alpha\in\mathrm{dom}(e)$ the definition of the $\leq^*$-order yields

$$(r\restriction\alpha)^\frown\vec{\pi}\Vdash_{\mathbb{P}\restriction\alpha} e(\alpha,\vec{\pi})\leq(c^\frown\vec{\sigma})(\alpha,\vec{\pi}\restriction\alpha),$$

for all $\vec{\pi}\in\mathrm{dom}(e(\alpha))$. Here $\vec{\pi}\restriction\alpha$ is a shorthand for the sequence of projections of the π_i's via $\pi_{\mathrm{mc}(a_i^r\cap\alpha),\mathrm{mc}(a_i^p\cap\alpha)}$. The above is equivalent to

$$(r\restriction\alpha)^\frown\vec{\pi}\Vdash_{\mathbb{P}\restriction\alpha} e(\alpha,\vec{\pi})\leq c(\alpha,\vec{\sigma}\restriction\alpha^\frown\vec{\pi}\restriction\alpha).$$

Notice that $(r\restriction\alpha)^\frown\vec{\pi}$ also forces $c(\alpha,\vec{\sigma}\restriction\alpha^\frown\vec{\pi}\restriction\alpha)\leq c(\alpha,\vec{\sigma}\restriction\alpha)$. Indeed,

$$(r\restriction\alpha)^\frown\vec{\pi}\leq(p\restriction\alpha)^\frown(\vec{\nu}\restriction\alpha^\frown\vec{\pi}\restriction\alpha)$$

and the latter forces the desired inequality (by definition of $\mathbb{M}$). Thereby, $(r\restriction\alpha)^\frown\vec{\pi}$ forces "$e(\alpha,\vec{\pi})\leq c(\alpha,\vec{\nu}\restriction\alpha)$". By a similar argument, this condition also forces "$e(\alpha,\vec{\pi})\leq d(\alpha,\vec{\eta}\restriction\alpha)$". Since this happens for an arbitrary $\vec{\pi}$ we may let one corresponding to the αth-projection of some $\vec{\tau}$ such that $r^\frown\vec{\tau}\in\bar{G}$. In particular, $(r\restriction\alpha)^\frown\vec{\pi}\in\bar{G}_\alpha$ and so $c(\alpha,\vec{\nu}\restriction\alpha)_{\bar{G}_\alpha}$ and $d(\alpha,\vec{\eta}\restriction\alpha)_{\bar{G}_\alpha}$ are compatible conditions in $\mathrm{Add}(\kappa^+,1)_{V[\bar{G}_\alpha]}$. □

Let $\vec{f}:=\langle f_\beta\mid\beta<\alpha\rangle$ be an injective enumeration of ${}^\kappa\kappa$ in $V[\bar{G}_\alpha]$. Define $\Phi\colon\alpha\to\kappa^+$ as $\Phi(\beta):=\gamma$ where γ is the least ordinal $<\kappa^+$ for which

$$c_\alpha(\gamma+\xi)=f_\beta(\xi),\ \text{for all}\ \xi<\kappa.$$

The above definition is run inside $V[G]$. If well-defined, Φ establishes an injection between α and κ^+. In particular, α is collapsed to κ^+.[8]

Let us thus show that Φ is well-defined. For each $\beta<\alpha$ consider

$$D_\beta:=\{(p,c)\in\mathbb{M}/\bar{G}\mid\exists\gamma<\kappa^+\,\forall\vec{\pi},\xi\,(c(\alpha,\vec{\pi})_{\bar{G}_\alpha}(\gamma+\xi)=f_\beta(\xi))\}.$$

Claim 2.16.2. *D_β is dense in $\mathbb{M}/\bar{G}$.*

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[6]By density, there is always $(p,c)\in G$ with $\alpha\in\mathrm{dom}(c)$. Thus, c_α is not $\varnothing$.

[7]Note that, the notation $\vec{\nu}\restriction\alpha$ is imprecise in that it does not exhibit the dependence on p. We warn our readers that during this proof $\vec{\nu}\restriction\alpha$ (resp. $\vec{\eta}\restriction\alpha$) is obtained using the αth projection associated to p (resp. q).

[8]Recall that κ^+ is preserved by $\mathbb{M}$.

Proof of claim. Let $(p, c) \in \mathbb{M}/\bar{G}$. Without loss of generality, $\alpha \in \mathrm{dom}(c)$. The definition of the forcing (see Definition 2.6(2)) implies that $c(\alpha, \vec{\pi})_{\bar{G}_\alpha} \in \mathrm{Add}(\kappa^+, 1)_{V[\bar{G}_\alpha]}$ for all $\vec{\pi} \in \mathrm{dom}(c(\alpha))$. Working in $V[\bar{G}_\alpha]$ let $\gamma < \kappa^+$ be above $\sup_{\vec{\pi} \in \mathrm{dom}(c(\alpha))} \mathrm{dom}(c(\alpha, \vec{\pi})_{\bar{G}_\alpha})$. This choice is possible because there are at most κ-many possible $\vec{\pi}$ and κ^+ is a regular cardinal in $V[\bar{G}_\alpha]$. Let $c^* := \{\langle \gamma + \xi, f_\beta(\xi)\rangle \mid \xi < \kappa\}$ and note that $c^* \cup c(\alpha, \vec{\pi})_{\bar{G}_\alpha}$ is a well-defined condition in $\mathrm{Add}(\kappa^+, 1)_{V[\bar{G}_\alpha]}$. Moreover, adding c^* to the $c(\alpha, \vec{\pi})_{\bar{G}_\alpha}$ does not conflict with the requirement that the conditions increase with larger $\vec{\pi}$ (Definition 2.6(3)). Let $\dot{c}^*$ be a $\mathbb{P} \restriction \alpha$-name such that $(\dot{c}^*)_{\bar{G}_\alpha} = c^*$. By modifying c^* is necessary we may assume that

$$\mathbb{1} \Vdash_{\mathbb{P}\restriction\alpha} \dot{c}^* \cup c(\alpha, \vec{\pi}) \in \dot{\mathrm{Add}}(\kappa^+, 1).^9$$

Let d be the function with $\mathrm{dom}(d) = \mathrm{dom}(c)$, $\mathrm{dom}(d(\alpha)) = \mathrm{dom}(c(\alpha))$ and for each $\vec{\pi} \in \mathrm{dom}(c(\alpha))$, $d(\alpha, \vec{\pi}) := \dot{c}^* \cup c(\alpha, \vec{\pi})$. It is clear that $(p, d) \in \mathbb{M}/\bar{G}$, $(p, d) \leq^* (p, c)$ and that it belongs to D_β. This shows that D_β is dense. □

Since G is also $\mathbb{M}/\bar{G}$-generic over $V[\bar{G}]$ we can let $(p, c) \in D_\beta \cap G$. It follows that for some γ, $c_\alpha(\gamma + \xi) = f_\beta(\xi)$ for all $\xi < \kappa$. So $\Phi(\beta)$ is defined. □

2.4. **Chain condition.** In this section we show that $\mathbb{M}$ does not collapse λ. This will be a consequence of the chain condition of $\mathbb{M}$. Since we are aiming for $\mathbb{M}$ to be the first step in an iteration *á-la-Σ-Prikry* we need to identify a stronger property which is iterable:

Definition 2.17. We say that $\mathbb{M}$ is *λ-Knaster to <λ-Linked* if for every $\mathcal{X} \in [\mathbb{M}]^\lambda$ there are $\mathcal{Y} \in [\mathcal{X}]^\lambda$, a regular cardinal $\delta < \lambda$ and a map

$$\mathfrak{C} \colon \{(p, c)^\frown \vec{\nu} \mid (p, c) \in \mathcal{Y},\, \vec{\nu} \in \textstyle\prod_{\ell(p) \leq n \leq k} A^p_n,\, k \geq \ell(p)\} \to H_\delta$$

such that the following holds:

$$(\dagger)\ \ \mathfrak{C}((p, c)^\frown \vec{\nu}) = \mathfrak{C}((q, d)^\frown \vec{\eta}) \Rightarrow \exists (r, d) \in \mathbb{M}\, ((r, d) \leq^* (p, c)^\frown \vec{\nu}, (q, d)^\frown \vec{\eta}).$$

The idea behind the previous concept is the following. Given a collection of λ-many conditions $\mathcal{X}$ in $\mathbb{M}$ one can shrink it to a set $\mathcal{Y}$ which admits a compatibility function $\mathfrak{C}$. The fact that $\mathfrak{C}$ is defined over the minimal extensions of $(p, c) \in \mathcal{Y}$ and not just on $\mathcal{Y}$ has to do with the iterability of this property. For more details see the proof of Lemma 3.16. The rest of this section will be devoted to show that, indeed, $\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked.

Lemma 2.18. *If $\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked then it is λ-Knaster.*

In particular, if $\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked then it forces "$\lambda = \kappa^{++}$".

Proof. Let $\mathcal{X} \in [\mathbb{M}]^\lambda$ and use that $\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked to find $\mathcal{Y} \in [\mathcal{X}]^\lambda$ and a compatibility function $\mathfrak{C}$ as above. Since $\mathrm{Im}(\mathfrak{C}) \subseteq H_\delta$ and λ is inaccessible we can find $\mathcal{Z} \subseteq \mathcal{Y}$ of size λ where $\mathfrak{C}$ is constant. □

[9] For instance, by taking τ a $\mathbb{P} \restriction \alpha$-name forced by $p \restriction \alpha$ to be $\dot{c}^*$ and forced to be $\varnothing$ by conditions that are incompatible with $p \restriction \alpha$.

Definition 2.19. Let $\delta \in (\kappa, \lambda)$ be inaccessible. For $(p, c) \in \mathbb{M} \restriction \delta$ define

$$\mathfrak{c}_\delta(p, c) := \langle p^\frown\vec{\nu}, \alpha, c(\alpha, \vec{\nu} \restriction \alpha) \mid \alpha \in \mathrm{dom}(c),\ \vec{\nu} \in [\textstyle\prod_{n \geq \ell(p)} A^p_n]^{<\omega}\rangle,$$

where $\vec{\nu} \restriction \alpha := \langle \pi_{\mathrm{mc}(a^p_i),\mathrm{mc}(a^p_i \cap \alpha)}(\nu_i) \mid \ell \leq i \leq k\rangle$.

Remark 2.20. $\mathfrak{c}_\delta$ has range in H_δ. Indeed, $p^\frown\vec{\nu} \in H_\delta$, $\mathrm{dom}(c) \in [\delta]^{\leq\kappa}$ and every $\mathbb{P} \restriction \alpha$-name $c(\alpha, \vec{\nu} \restriction \alpha)$ can be coded inside H_δ because $\mathbb{P} \restriction \alpha \in H_\delta$.

Lemma 2.21. *$\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked.*

Proof. Let $\{(p_\xi, c_\xi) \mid \xi < \lambda\} \subseteq \mathbb{M}$. Following Gitik [Git10, Lemma 2.15] one may assume that for each $\xi < \lambda$ the following hold:

- $\ell(p_\xi) = \ell$;
- $f^{p_\xi}_n \cup f^{p_\zeta}_n$ is a function for all $n < \omega$;
- for each $n \geq \ell$, $\{\mathrm{dom}(f^{p_\xi}_n) \cup a^{p_\xi}_n \mid \xi < \lambda\}$ forms a Δ-system with root Δ_n satisfying that $\mathrm{dom}(f^{p_\xi}_n) \cap a^{p_\zeta}_n = \emptyset$.

In addition, we may assume that

(1) for each $n \geq \ell$ and $\xi < \lambda$, $A^{p_\xi}_n = A_n$,
(2) for each $n \geq \ell$, $\{\pi_{\mathrm{mc}(a^{p_\xi}_n),*} \restriction (\Delta_n \cap a^{p_\xi}_n) \times A_n \mid \xi < \lambda\}$ is a singleton.[10]

Claim 2.21.1. *For each $\xi, \zeta < \lambda$, p_ξ and p_ζ are $\leq^*$-compatible.*
Moreover, for each $\vec{\nu} \in [\prod_{\ell \leq n} A_n]^{<\omega}$, $p_\xi^\frown\vec{\nu}$ and $p_\zeta^\frown\vec{\nu}$ are $\leq^*$-compatible.

Proof. The assertion that p_ξ and p_ζ are $\leq^*$-compatible is immediate. For the moreover claim note that the incompatibilities between $p_\xi^\frown\vec{\nu}$ and $p_\zeta^\frown\vec{\nu}$ must arise in the form of discrepancies between $\pi_{\mathrm{mc}(a^{p_\xi}_n),\delta}(\nu_n)$ and $\pi_{\mathrm{mc}(a^{p_\zeta}_n),\delta}(\nu_n)$ for δ's in Δ_n. However, these two values are the same by (2) above. $\square$

For each $\xi < \lambda$, $\mathrm{dom}(c_\xi)$ is a subset of λ of cardinality $\leq\kappa$. So, by further shrinking the indexing set, we may assume that $\{\mathrm{dom}(c_\xi) \mid \xi < \lambda\}$ is a Δ-system; say, with root Δ. Let $\delta \in (\kappa, \lambda)$ be an inaccessible cardinal with $\Delta \subseteq \delta$. By the forthcoming Lemma 2.22, $\{(p_\xi \restriction \delta, c_\xi \restriction \delta) \mid \alpha < \lambda\}$ are conditions in $\mathbb{M} \restriction \delta$. For each $\alpha < \lambda$ define

$$\mathfrak{C}((p_\xi, c_\xi)^\frown\vec{\nu}) := \langle \vec{\nu}, \mathfrak{c}_\delta(p_\xi \restriction \delta, c_\xi \restriction \delta)\rangle,$$

where $\mathfrak{c}_\delta$ is as in Definition 2.19. Clearly, $\mathfrak{C}$ has range in H_δ.

We claim that $\mathfrak{C}$ satisfies equation (†) of Definition 2.17. Suppose that $\mathfrak{C}((p_\xi, c_\xi)^\frown\vec{\nu})) = \mathfrak{C}((p_\zeta, c_\zeta)^\frown\vec{\eta}))$. Then $\vec{\nu} = \vec{\eta}$ and by Claim 2.21.1 both $p_\xi{}^\frown\vec{\nu}$ and $p_\zeta{}^\frown\vec{\nu}$ are $\leq^*$-compatible. Let r be $\leq^*$-stronger and d be the function with domain $\mathrm{dom}(c_\xi) \cup \mathrm{dom}(c_\zeta)$ such that

$$\forall \vec{\pi} \in \mathrm{dom}(d(\alpha)) = [\textstyle\prod_{n \geq \ell(r)} \pi_{\mathrm{mc}(a^r_n),\mathrm{mc}(a^r_n \cap \alpha)} ``A^r_n]^{<\omega},$$

for each $\alpha \in \mathrm{dom}(d) \setminus \Delta$ (i.e., outside the common domain),

$$d(\alpha, \vec{\pi}) := \begin{cases} (c_\xi{}^\frown\vec{\nu})(\alpha, \vec{\pi}_{\xi,\alpha}), & \text{if } \alpha \in \mathrm{dom}(c_\xi); \\ (c_\zeta{}^\frown\vec{\nu})(\alpha, \vec{\pi}_{\zeta,\alpha}) & \text{if } \alpha \in \mathrm{dom}(c_\zeta). \end{cases}$$

[10] Here $\pi_{\mathrm{mc}(a^{p_\xi}_n),*}$ denotes the map defined as $(\delta, \nu) \mapsto \pi_{\mathrm{mc}(a^{p_\xi}_n),\delta}(\nu)$.

If $\alpha \in \Delta$, $d(\alpha, \vec{\pi}) := (c_\xi{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\xi,\alpha})$. Above we used the following notation:

$$\vec{\pi}_{x,\alpha} := \langle \pi_{\mathrm{mc}(a_i^r \cap \alpha), \mathrm{mc}(a_i^{p_x} \cap \alpha)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle \text{ for } x \in \{\xi, \zeta\}.$$

Clearly (r, d) is a condition in $\mathbb{M}$ and $(r, d) \leq^* (p_\xi, c_\xi)^\frown \vec{\nu}$.

The reason for why (r, d) is $\leq^*$-stronger than $(p_\zeta, c_\zeta)^\frown \vec{\nu}$ is that

$$(c_\xi{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\xi,\alpha}) = (c_\zeta{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\zeta,\alpha})$$

whenever $\alpha \in \Delta$.

To see this, fix $\vec{\pi} \in \mathrm{dom}(d(\alpha))$ and let $\vec{\sigma}$ be a preimage of it in the measure one sets of r. Then,

$$(r, d)^\frown \vec{\sigma} \leq^* (p_\xi, c_\xi)^\frown \vec{\nu}^\frown \vec{\sigma}_\xi, (p_\zeta, c_\zeta)^\frown \vec{\nu}^\frown \vec{\sigma}_\zeta,$$

where $\vec{\sigma}_\xi$ (resp. $\vec{\sigma}_\zeta$) is a shorthand for $\vec{\sigma}_{\xi,\lambda}$ (resp. $\vec{\sigma}_{\zeta,\lambda}$). In particular, the same holds when projecting down to δ. Let $\vec{\nu}(\xi), \vec{\sigma}(\xi)$ be the sequence defined by the projections

$$\pi_{\mathrm{mc}(a_i^{p_\xi}), \mathrm{mc}(a_i^{p_\xi} \cap \delta)}(\nu_i) \text{ and } \pi_{\mathrm{mc}(a_i^{p_\xi}), \mathrm{mc}(a_i^{p_\xi} \cap \delta)}(\sigma_\xi(i)).$$

Define $\vec{\nu}(\zeta)$ and $\vec{\sigma}(\zeta)$ similarly using the projections associated to p_ζ.

Since $\mathfrak{c}_\delta(p_\xi \restriction \delta, c_\xi \restriction \delta) = \mathfrak{c}_\delta(p_\zeta \restriction \delta, c_\zeta \restriction \delta)$ it follows, by Definition 2.19, that $(p_\xi \restriction \delta)^\frown \langle \vec{\nu}(\xi)^\frown \vec{\sigma}(\xi) \rangle = q$, where q is: 1) a minimal extension of $p_\zeta \restriction \delta$; 2) compatible with $(p_\xi \restriction \delta)^\frown \langle \vec{\nu}(\xi)^\frown \vec{\sigma}(\xi) \rangle$; and 3), has the same length as $(p_\xi \restriction \delta)^\frown \langle \vec{\nu}(\xi)^\frown \vec{\sigma}(\xi) \rangle$. It must be the case that

$$(p_\xi \restriction \delta)^\frown \langle \vec{\nu}(\xi)^\frown \vec{\sigma}(\xi) \rangle = (p_\zeta \restriction \delta)^\frown \langle \vec{\nu}(\zeta)^\frown \vec{\sigma}(\zeta) \rangle.$$

Using again Definition 2.19 and the commutativity of the projections one can check that $(c_\xi{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\xi,\alpha}) = (c_\zeta{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\zeta,\alpha})$. □

2.5. **Projections.** In this section we show that there is a natural projection between $\mathbb{M}$ and $\mathbb{M} \restriction \alpha$ for all regular cardinals α in $[\kappa^+, \lambda)$ (i.e., $\alpha \in \mathcal{R}$). This will be instrumental in our proof of the ineffable tree property at λ.

Lemma 2.22. *For each $\alpha \in \mathcal{R}$ the map $(p, c) \mapsto (p \restriction \alpha, c \restriction \alpha)$ defines a length-preserving projection from $\mathbb{M}$ to $\mathbb{M} \restriction \alpha$.*

Proof. First, note that $c \restriction \alpha$ has the right shape and thus $(p \restriction \alpha, c \restriction \alpha) \in \mathbb{M} \restriction \alpha$. Indeed, this follows from commutativity of the projections; namely,

$$\pi_{\mathrm{mc}(a_i^p), \mathrm{mc}(a_i^p \cap \beta)} \text{“} A_i^p = \pi_{\mathrm{mc}(a_i^p \cap \alpha), \mathrm{mc}(a_i^p \cap \beta)} \text{“} A_i^{p \restriction \alpha}.$$ [11]

Suppose $(p, c) \leq (q, d)$ and let $\vec{\nu}$ be with $(p, c) \leq^* (q, d)^\frown \vec{\nu}$. We show that

$$(p \restriction \alpha, c \restriction \alpha) \leq^* (q \restriction \alpha, d \restriction \alpha)^\frown \langle \pi_{\mathrm{mc}(a_i^q), \mathrm{mc}(a_i^q \cap \alpha)}(\nu_i) \mid i \leq |\vec{\nu}| \rangle.$$

For this it suffices to check that

$$(p \restriction \beta)^\frown \vec{\pi} \Vdash_{\mathbb{P} \restriction \beta} c(\beta, \vec{\pi}) \leq (d^\frown \vec{\nu})(\beta, \langle \pi_{\mathrm{mc}(a_i^p \cap \beta), \mathrm{mc}(a_i^q \cap \beta)}(\pi_i) \mid \ell(p) \leq i \leq k \rangle)$$

[11] $A_i^{p \restriction \alpha}$ is by definition $\pi_{\mathrm{mc}(a_i^p), \mathrm{mc}(a_i^p \cap \alpha)} \text{“} A_i^p$.

for all $\beta \in \mathrm{dom}(d) \cap \alpha$ and $\vec{\pi} \in \prod_{\ell(p) \leq i \leq k} \pi_{\mathrm{mc}(a_i^p \cap \alpha), \mathrm{mc}(a_i^p \cap \beta)}$ "$A_i^{p \restriction \alpha}$. However, this happens automatically. Indeed, it follows from $(p,c) \leq^* (q^\frown \vec{\nu}, d^\frown \vec{\nu})$ (see Clause (III) of Definition 2.6).

Let us finally check that the map is a projection. Let $(q,d) \leq (p \restriction \alpha, c \restriction \alpha)$ in $\mathbb{M} \restriction \alpha$. First, let $r \leq p$ be such that $r \restriction \alpha \leq^* q$. Say $\vec{\nu}$ witnesses that $r \leq^* p^\frown \vec{\nu}$. Define the *Mitchell-part* of r as follows. Let e be the function with domain $\mathrm{dom}(e) := \mathrm{dom}(d) \cup (\mathrm{dom}(c) \setminus \alpha)$ such that for each $\beta \in \mathrm{dom}(e)$:

- $\mathrm{dom}(e(\beta)) := [\prod_{i \geq \ell(r)} \pi_{\mathrm{mc}(a_i^r), \mathrm{mc}(a_i^r \cap \beta)}$ "$A_i^r]^{<\omega}$;
- for each $\vec{\pi} = \langle \pi_{\ell(r)}, \ldots, \pi_k \rangle \in \mathrm{dom}(e(\beta))$,

$$e(\beta, \vec{\pi}) := \begin{cases} (c^\frown \vec{\nu})(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle), & \text{if } \beta \notin \mathrm{dom}(d); \\ d(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^q \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle), & \text{if } \beta \in \mathrm{dom}(d). \end{cases}$$

It is clear that $\mathbb{1}_{\mathbb{P} \restriction \beta}$ forces $e(\beta, \vec{\pi})$ to be a condition in $\mathrm{Add}(\kappa^+, 1)$ and that $e(\beta, \vec{\pi})$ becomes stronger with longer $\vec{\pi}$'s. Namely, $(r,e) \in \mathbb{M}$.

Let us check that $(r,e) \leq (p,c)$ and that $(r \restriction \alpha, e \restriction \alpha) \leq^* (q,d)$. Looking at Definition 2.6(III), the latter is pretty obvious. The former also holds in that $(r,e) \leq^* (p,c)^\frown \vec{\nu}$. To see this, note that for $\beta \in \mathrm{dom}(c) \setminus \alpha$ the verification is obvious. For $\beta \in \mathrm{dom}(d)$ and $\vec{\pi} = \langle \pi_{\ell(r)}, \ldots, \pi_k \rangle$ in $\prod_{\ell(r) \leq i \leq k} \pi_{\mathrm{mc}(a_i^r), \mathrm{mc}(a_i^r \cap \beta)}$ "A_i^r we need to check that

$$(r \restriction \beta)^\frown \vec{\pi} \Vdash_{\mathbb{P} \restriction \beta} e(\beta, \vec{\pi}) \leq (c^\frown \vec{\nu})(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle).$$

To argue this we will use that $(q,d) \leq (p \restriction \alpha, c \restriction \alpha)$. Let $\vec{\sigma}$ be such that $(q,d) \leq^* (p \restriction \alpha, c \restriction \alpha)^\frown \vec{\sigma}$. Necessarily $\vec{\sigma} = \langle \pi_{\mathrm{mc}(a_i^p), \mathrm{mc}(a_i^p \cap \alpha)}(\nu_i) \mid i < |\vec{\nu}| \rangle$. By definition we have that $(q \restriction \beta)^\frown \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^q \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle$ forces

$$d(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^q \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle) \leq$$
$$((c \restriction \alpha)^\frown \vec{\sigma})(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi) \mid \ell(r) \leq i \leq k \rangle).$$

Since $\vec{\sigma}$ and $\vec{\nu}$ project the same way below α a moment of reflection makes clear that $((c \restriction \alpha)^\frown \sigma)(\beta, \cdot) = (c^\frown \vec{\nu})(\beta, \cdot)$. Thus, the above condition forces

$$d(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^q \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle) \leq$$
$$(c^\frown \vec{\nu})(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi) \mid \ell(r) \leq i \leq k \rangle).$$

Since $(r \restriction \beta)^\frown \vec{\pi} \leq (q \restriction \beta)^\frown \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^q \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle$ it follows that $(r \restriction \beta)^\frown \vec{\pi} \leq (q \restriction \beta)$ forces the desired property. □

Let us close this section analyzing the subforcing of $\mathbb{M}$ where the Prikry part is fixed. Formally, let π be the natural projection between $\mathbb{M}$ and $\mathbb{P}$; namely, $\pi \colon (p,c) \mapsto p$. For each $n < \omega$ let us denote by $\mathbb{M}_n^\pi$ the subforcing of $\mathbb{M}_n := \{(p,c) \in \mathbb{M} \mid \ell(p) = n\}$ endowed with the following order:

$$(p,c) \leq^\pi (q,d) \text{ iff } (p,c) \leq^* (q,d) \text{ and } p = q.$$

Lemma 2.23. *For each $n < \omega$, $\mathbb{M}_n^\pi$ is κ^+-directed-closed.*

Proof. Let $D \subseteq \mathbb{M}_n^\pi$ be a $\leq^\pi$-directed set of conditions of size $\leq \kappa$. Let p be the common EBPF-part of conditions in D. Let c be the function with

- $\mathrm{dom}(c) := \bigcup_{(p,d)\in D} \mathrm{dom}(d)$,
- $\mathrm{dom}(c(\alpha)) := [\prod_{n\geq \ell(p)} \pi_{\mathrm{mc}(a^p_n),\mathrm{mc}(a^p_n\cap\alpha}\text{“}A^p_n]^{<\omega}$,
- $c(\alpha,\vec{\pi}) := \bigwedge\{d(\alpha,\vec{\pi}) \mid \alpha\in \mathrm{dom}(d)\}$,

where this lower bound is forced by $(p\restriction\alpha)^\frown\vec{\pi}$ to be such. As usual, we tweak $c(\alpha,\vec{\pi})$ a bit so that $\mathbb{1}_{\mathbb{P}\restriction\alpha}$ forces it to be a condition $\dot{\mathrm{Add}}(\kappa^+,1)$. Clearly, (p,c) is a $\leq^\pi$-lower bound for D. □

2.6. **Conclusion.** With all the previous results at hand it is easy to verify that $\mathbb{M}$ satisfies all the axioms of the Σ-Prikry framework (see [PRS22, §2]) with the only exception of the existence of a compatibility function c. Let us call this family *weak-Σ-Prikry*. Remember that in the current scenario the preservation of cardinals $\geq\lambda$ is handled by Lemma 2.21.

The following is a summary of the main properties of $\mathbb{M}$.

Theorem 2.24. *The following properties are true for $\mathbb{M}$:*

(1) $\mathbb{M}$ is weak-Σ-Prikry having property $\mathcal{D}$;
(2) $\mathbb{M}$ forces $2^\kappa=\lambda$ and preserves cardinals except those in (κ^+,λ).
(3) For each V-regular cardinal $\alpha\in[\kappa^+,\lambda)$ the map $(p,c)\mapsto(p\restriction\alpha,c\restriction\alpha)$ defines a length-preserving projection from $\mathbb{M}$ to $\mathbb{M}\restriction\alpha$.
(4) $\mathbb{M}$ forces the following: Given $\langle S_n\mid n\leq m\rangle$ finitely-many stationary subsets of $\{\alpha<\kappa^+\mid \mathrm{cf}(\alpha)^V\neq\omega\}$ there is $\alpha<\kappa^+$ with $\mathrm{cf}(\alpha)>\omega$ such that each $S_n\cap\alpha$ is stationary in α.
(5) $\mathbb{M}$ forces $\mathrm{ITP}(\lambda)$.

To establish (4) one needs each κ_n to be Laver-indestructible as in those circumstances one can appeal to [PRS19, Corollary 5.10]. Similarly, (5) requires λ to be supercompact (the proof of this latter fact is postponed to §5 where a more general result is proved).

3. Killing a fragile stationary set

In this section we describe a forcing poset which given a weak-Σ-Prikry forcing $\mathbb{Q}$ and a *fragile stationary set* $\dot{T}$ returns a weak-Σ-Prikry forcing $\mathbb{A}:=\mathbb{A}(\mathbb{Q},\dot{T})$ killing the stationarity of $\dot{T}$ and projecting onto $\mathbb{Q}$. The above-mentioned poset $\mathbb{A}$ will be a variation of the Sharon of [PRS22]. The modification we present here will secure the existence of projections from $\mathbb{A}$ onto $\mathbb{A}\restriction\alpha$, for all $\alpha\in[\kappa^+,\lambda)$ inaccessible. For simplicity, denote by $\mathcal{R}'$ the set of all inaccessible cardinals in $[\kappa^+,\lambda)$.

We present the arguments for a general $\mathbb{Q}$ to use the obtained results at any successor stage of the eventual iteration. Thus, it is helpful to think of $\mathbb{Q}$ as a stage of a Σ-Prikry-styled iteration starting with the poset $\mathbb{M}$ of §2.1.

Setup 3. We are given $\langle\mathbb{Q}\restriction\alpha\mid\alpha\in\mathcal{I}\rangle$ and $\langle\cdot\restriction\alpha\mid\alpha\in\mathcal{I}\rangle$ such that:

(1) $\mathcal{I}\subseteq\mathcal{R}'\cup\{\lambda\}$ is co-bounded in λ, with $\lambda\in\mathcal{I}$;
(2) $\mathbb{Q}\restriction\alpha$ is a weak-Σ-Prikry forcing having property $\mathcal{D}$ and $\mathbb{Q}\restriction\alpha\subseteq H_\lambda$;
(3) $\mathbb{Q}\restriction\alpha$ is λ-Knaster to $<\lambda$-Linked (see Definition 2.17);

(4) $\mathbb{Q}\restriction\alpha$ projects to $\mathbb{M}\restriction\alpha$;
(5) $\cdot\restriction\alpha\colon \mathbb{Q}\to\mathbb{Q}\restriction\alpha$ is a length-preserving projection, where $\mathbb{Q}:=\mathbb{Q}\restriction\lambda$;
(6) for each $p\in\mathbb{Q}$, for all large $\alpha<\lambda$, $p=p\restriction\alpha$.

Let $r^\star\in\mathbb{Q}$ and $\dot{T}$ be a $\mathbb{Q}$-name for a $r^\star$-fragile stationary subset of κ^+ (Definition 6.1 in [PRS23]). By definition of fragility, for all $q\leq r^\star$,

$$q\Vdash_{\mathbb{Q}_{\ell(q)}} \text{“}\dot{T}_{\ell(q)}\text{ is nonstationary”}.$$

Thus, for each $n\geq\ell(r^*)$, let a $\mathbb{Q}_n$-name $\dot{C}_n$ for a club subset of κ^+ such that for all $q\leq r^\star$, $q\Vdash_{\mathbb{Q}_{\ell(q)}}\dot{T}_{\ell(q)}\cap\dot{C}_{\ell(q)}=\emptyset$. Using that $\mathbb{Q}$ is λ-cc (Clause (3)) there is an inaccessible cardinal $\delta<\lambda$ such that $\dot{T}$ and $\dot{C}_n$ are, respectively, $\mathbb{Q}\restriction\alpha$ and $(\mathbb{Q}\restriction\alpha)_n$-names for all $\alpha\in\mathcal{I}\setminus\delta$.

For each $\alpha\in\mathcal{I}\setminus\delta$ define the following binary relation:

$$R\restriction\alpha:=\{(\varrho,q)\in\mu\times\mathbb{Q}\restriction\alpha\mid \forall r\leq_{\mathbb{Q}\restriction\alpha} q\restriction\alpha\ (r\Vdash_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}}\varrho\in\dot{C}_{\ell(r)})\}.$$

$R\restriction\alpha$ is downwards closed; namely, for all $(\varrho,q)\in R\restriction\alpha$

$$q'\leq_{\mathbb{Q}\restriction\alpha} q\ \Rightarrow\ (\varrho,q')\in R\restriction\alpha.$$

As in [PRS23, §6], let us define a $\mathbb{Q}$-name

$$\dot{T}^+:=\{(\check{\varrho},p)\mid(\varrho,p)\in\kappa^+\times\mathbb{Q}\ \wedge\ \forall q\leq p\,(q\Vdash_{\mathbb{Q}_{\ell(q)}}\check{\varrho}\notin\dot{C}_{\ell(q)})\}.$$

By [PRS22, Lemma 4.6] the trivial condition of $\mathbb{Q}$ forces $\dot{T}\subseteq\dot{T}^+$ (hence $\dot{T}^+$ is stationary) and if $(\varrho,q)\in R\restriction\lambda$ then $q\Vdash_{\mathbb{Q}}\varrho\notin\dot{T}^+$. Also, by λ-ccness of $\mathbb{Q}$, the $\mathbb{Q}$-name $\dot{T}^+$ can be identified with a $\mathbb{Q}\restriction\alpha$-name, for all large $\alpha<\lambda$. Thus, if $(\varrho,q)\in R\restriction\alpha$ then $q\Vdash_{\mathbb{Q}\restriction\alpha}\varrho\notin\dot{T}^+$. By shrinking $\mathcal{I}\setminus\delta$ we ensure this configuration holds for every $\alpha\in\mathcal{I}\setminus\delta$. Denote $\mathcal{J}:=\mathcal{I}\setminus\delta$.

Notation 3.1. Given $p\in\mathbb{Q}$ we denote by A^p_i's the measure one sets of the EBPF-part of p – denote this latter $\mathsf{EBPF}(p)$. Note that this makes sense because $\mathbb{Q}$ projects to $\mathbb{M}$ (Clause (4)) which certainly has an EBPF-part.

For $\vec{\nu}\in\prod_{\ell(p)\leq i\leq k}A^p_i$, we denote by $p^\frown\vec{\nu}$ the weakest extension of p whose EBPF-part is given by $\mathsf{EBPF}(p)^\frown\vec{\nu}$. We also denote the corresponding EBPF-projections by $\pi_{\mathrm{mc}(a^p_i),\mathrm{mc}(a^p_i\cap\alpha)}$.

Definition 3.2. Let $p\in\mathbb{Q}$. A function S is a *$\mathcal{J}$-p-labeled tree* if $\mathrm{dom}(S)\in[\mathcal{J}]^{\leq\kappa}$ and for each $\alpha\in\mathcal{J}$ the following hold:

(1) $S(\alpha)\colon\mathrm{dom}(S(\alpha))\to\{C\subseteq\kappa^+\mid C\text{ closed and bounded}\}$ where
$$\mathrm{dom}(S(\alpha)):=\{\langle\vec{\pi},t\rangle\mid\vec{\pi}\in\textstyle\prod_{\ell(p)\leq n\leq k}\pi_{\mathrm{mc}(a^p_i)\,\mathrm{mc}(a^p_i\cap\alpha)}\text{“}A^p_i\}.$$
(2) for all $\vec{\pi}\in\mathrm{dom}(S(\alpha))$,
$$p\restriction\alpha^\frown\vec{\pi}\Vdash_{\mathbb{Q}\restriction\alpha}S(\alpha,\vec{\pi})\cap\dot{T}^+=\emptyset.$$
(3) if $\vec{\pi},\vec{\sigma}\in\mathrm{dom}(S(\alpha))$ and $p\restriction\alpha^\frown\vec{\sigma}\leq p\restriction\alpha^\frown\vec{\pi}$ then
$$S(\alpha,\vec{\pi})\subseteq S(\alpha,\vec{\sigma}).$$

(4) there is $m(S) < \omega$, called the *delay* for S, such that, for each $\alpha \in \mathcal{J}$, if $p \restriction \alpha^\frown \vec{\sigma} \leq p \restriction \alpha^\frown \vec{\pi}$ and $\ell(p \restriction \alpha^\frown \vec{\pi}) \geq \ell(p \restriction \alpha) + m(S)$ then $(\max(S(\alpha, \vec{\sigma})), p \restriction \alpha^\frown \vec{\pi}) \in R \restriction \alpha$.

Remark 3.3. For each $\alpha \in \mathcal{J}$, $S(\alpha, \cdot)$ is a $p \restriction \alpha$-labeled tree in the sense of [PRS22] with respect to the binary relation $R \restriction \alpha$. Therefore, the above is just a multidimensional version of the notion considered in [PRS22], with a uniform delay. The common delay will help us define the type map in Definition 3.19, which is needed to get the Prikry property of $\mathbb{A}$. The requirement $\mathrm{dom}(S) \in [\mathcal{J}]^{\leq \kappa}$ is used in Lemma 3.8 and Fact 3.21.

Definition 3.4. A sequence $\vec{S} = \langle S_i \mid i \leq \gamma^{\vec{S}} \rangle$ of $\mathcal{J}$-p-labeled trees is called a *$\mathcal{J}$-p-strategy* if $\gamma^{\vec{S}} < \kappa^+$, $\mathrm{dom}(S_i) = \mathrm{dom}(S_0)$ for all $i \leq \gamma^{\vec{S}}$, and for each $\alpha \in \mathrm{dom}(S_0)$, $\vec{S}(\alpha) := \langle S_i(\alpha) \mid i \leq \gamma^{\vec{S}} \rangle$ is a $p \restriction \alpha$-strategy in the sense of [PRS22].

Let us now present our new Sharon-like poset:

Definition 3.5. For each $\alpha \in \mathcal{J}$, let $\mathbb{A} \restriction \alpha := \mathbb{A}(\mathbb{Q} \restriction \alpha, \mathcal{J} \cap \alpha^+, \dot{T}^+)$ be the poset consisting of pairs $(p, \vec{S})$ such that the following hold:

(1) $p \in \mathbb{Q} \restriction \alpha$;
(2) $\vec{S}$ is either empty or a $(\mathcal{J} \cap \alpha^+)$-$p \restriction \alpha$-strategy.

Write $(q, \vec{Q}) \leq (p, \vec{S})$ if and only if $q \leq p$, $\gamma^{\vec{Q}} \geq \gamma^{\vec{S}}$, $\mathrm{dom}(S_0) \subseteq \mathrm{dom}(Q_0)$ and for each $i \leq \gamma^{\vec{S}}$, $\beta \in \mathrm{dom}(S_i)$ and $\langle \vec{\pi}, t \rangle \in \mathrm{dom}(Q_i(\beta))$,

$$Q_i(\beta, \vec{\pi}, t) = S_i(\beta, \langle \pi_{\mathrm{mc}(a_i^q \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi_i) \mid \ell \leq i \leq \ell + |\vec{\pi}| \rangle),$$

Given $(p, \vec{S})$ and $\vec{\nu}$ as above, define $(p, \vec{S})^\frown \vec{\nu} := (q, \vec{Q})$ as follows:

- $q = p^\frown \vec{\nu}$;
- $\vec{Q} := \langle Q_i \mid i \leq \gamma^{\vec{S}} \rangle$ where:
  - $\mathrm{dom}(Q_i) := \mathrm{dom}(S_i)$ and $\mathrm{dom}(Q_i(\beta))$ being $\prod_{\ell(q) \leq n \leq k} \pi_{\mathrm{mc}(a_i^q)\,\mathrm{mc}(a_i^q \cap \beta)}$ "A_i^q.
  - The value of $Q_i(\beta, \vec{\pi})$ is

$$S_i(\beta, \langle \pi_{\mathrm{mc}(a_i^q \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi_i) \mid \ell(q) \leq i \leq k \rangle).$$

Remark 3.6. It is routine to verify that $(p, \vec{S})^\frown \vec{\nu} \leq (p, \vec{S})$. In fact, the former is the weakest extension of (p, S) with first coordinate $p^\frown \vec{\nu}$.

Our main forcing will be $\mathbb{A} := \mathbb{A} \restriction \lambda$.

3.1. **Projections.**

Lemma 3.7. *For each $\alpha \in \mathcal{J}$ there exists a length-preserving projection $\restriction \alpha \colon \mathbb{A} \to \mathbb{A} \restriction \alpha$ given by $(p, \vec{S}) \mapsto (p \restriction \alpha, \langle S_i \restriction \alpha^+ \mid i \leq \gamma^{\vec{S}} \rangle)$.*

Proof. A moment of reflection makes clear that $\cdot \restriction \alpha$ is well-defined and order-preserving. Suppose $(q, \vec{Q}) \leq (p \restriction \alpha, \vec{S} \restriction \alpha)$ and let $r \leq p$ be such that

$r \restriction \alpha \leq^* q$. This $r \in \mathbb{Q}$ exists by (5) in Setup 3. Define $\vec{R} := \langle R_i \mid i \leq \gamma^{\vec{Q}} \rangle$ where each R_i has domain $\mathrm{dom}(Q_i) \cup (\mathrm{dom}(S_i) \setminus \alpha^+)$ and $\mathrm{dom}(R_i(\beta))$ is

$$\prod_{\ell(r) \leq n \leq k} \pi_{\mathrm{mc}(a_i^r)\,\mathrm{mc}(a_i^r \cap \beta)} \text{``} A_i^r.$$

For each $\beta \in \mathrm{dom}(R_i)$ and $\vec{\pi} \in \mathrm{dom}(R_i(\beta))$, $R_i(\beta, \vec{\pi})$ equals

$$\begin{cases} S_{\min\{i, \gamma^{\vec{S}}\}}(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi_i) \mid \ell(p) \leq i \leq k \rangle), & \text{if } \beta \geq \alpha^+; \\ Q_i(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^q \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle), & \text{if } \beta < \alpha^+. \end{cases}$$

It is routine to check that $(r, \vec{R})$ is a well-defined condition in $\mathbb{A}$ and that $(r, \vec{R}) \leq (p, \vec{S})$ and $(r, \vec{R}) \restriction \alpha \leq^* (q, \vec{Q})$. The key point is that for each $\beta, \beta' \in \mathrm{dom}(R_i)$ the functions $R_i(\beta, \cdot)$ and $R_i(\beta', \cdot)$ are independent. $\square$

Lemma 3.8. *$\mathbb{A} \subseteq H_\lambda$ and for each $(p, \vec{S}) \in \mathbb{A}$ there are co-boundedly many $\alpha < \lambda$ such that $(p, \vec{S}) \restriction \alpha = (p, \vec{S})$.*

Proof. This is one reason for requiring $\mathrm{dom}(S_i) \in [\mathcal{J}]^{\leq \kappa}$ in the definition of $\mathcal{J}$-p-strategy. Simply let some $\alpha < \lambda$ such that $p \restriction \alpha = p$ above $\sup(\mathrm{dom}(S_0)) < \lambda$ (by Clause (6) in page 16). Thus, $(p, \vec{S}) \restriction \alpha = (p, \vec{S})$. $\square$

Let us argue that $\mathbb{A}$ is weak-Σ-Prikry with property $\mathcal{D}$. The same is true for the trucations $\mathbb{A} \restriction \alpha$ and $\mathbb{Q} \restriction \alpha$ modulo obvious changes.

Definition 3.9 (Maps)**.**

(1) For $(p, \vec{S}) \in \mathbb{A}$ and $q \leq p$ define $\pitchfork(p, \vec{S})(q) := (q, \vec{Q})$ where:
- $\vec{Q} := \langle Q_i \mid i \leq \gamma^{\vec{S}} \rangle$.
- $\mathrm{dom}(Q_i) := \mathrm{dom}(S_i)$ and $\mathrm{dom}(Q_i(\beta))$ consists of sequences $\vec{\pi}$, such that for some $k < \omega$, $\vec{\pi} \in \prod_{\ell(q) \leq n \leq k} \pi_{\mathrm{mc}(a_i^q)\,\mathrm{mc}(a_i^q \cap \beta)} \text{``} A_i^q$.
- The value of $Q_i(\beta, \vec{\pi})$ is

$$S_i(\beta, \langle \pi_{\mathrm{mc}(a_i^q \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi_i) \mid \ell(q) \leq i \leq k \rangle).$$

The next lemma can be proved exactly as in Lemmas 6.9 and 6.10 from [PRS22] noting that for each $\alpha \in \mathcal{J}$, $\langle S_i(\alpha) \mid i \leq \gamma^{\vec{S}} \rangle$ is as in [PRS22].

Lemma 3.10. *$(\pitchfork, \pi)$ defines a forking projection between $\mathbb{A}$ and $\mathbb{Q}$.*

By the results in [PRS22, §2] we infer:

Corollary 3.11. *$(\mathbb{A}, \vec{\varsigma})$ is weak-Σ-Prikry having property $\mathcal{D}$.*

Remark 3.12. The natural modification of the pair $(\pitchfork, \pi)$ yields a forking projection $(\pitchfork^\alpha, \pi^\alpha)$ between $\mathbb{A} \restriction \alpha$ and $\mathbb{Q} \restriction \alpha$.

3.2. **Chain condition.** In this section we show that $\mathbb{A} \restriction \alpha$ is λ-Knaster to $<\lambda$-Linked, hence verifying Clause (3) of Setup 3. For this purpose we need to introduce an auxiliary property which enables *lifting* the λ-Knaster to $<\lambda$-Linkedness of $\mathbb{Q} \restriction \alpha$ to $\mathbb{A} \restriction \alpha$.

Notation 3.13. For a condition p in $\mathbb{Q}$ or in $\mathbb{A}$,

$$W(p) := \{p^\frown \vec{\nu} \mid \vec{\nu} \in \prod_{\ell(p) \leq i \leq k} A_i^p, k < \omega\}.$$

Namely, $W(p)$ gathers all the weakest extensions of p.

Definition 3.14. Assume that $(\pitchfork, \pi)$ is a forking projection from $\mathbb{A}$ to $\mathbb{Q}$. We say that $(\pitchfork, \pi)$ *is a compatibility forking projection* (c-forking projection for short) if the following is true: For each $\mathcal{X} \in [\mathbb{A}]^\lambda$ if $\mathcal{Y} \in [\mathbb{Q}]^\lambda$ and $\mathfrak{C}_\mathbb{Q} \colon \bigcup_{p \in \mathcal{Y}} W(p) \to H_\delta$ are such that

$$\mathfrak{C}_\mathbb{Q}(q) = \mathfrak{C}_\mathbb{Q}(r) \Rightarrow \exists s \in \mathbb{Q}\, (s \leq^* q, r),$$

setting $\mathcal{Z} := \{a \in \mathcal{X} \mid \pi(a) \in \mathcal{Y}\}$, there is $\delta^* \in [\delta, \lambda)$ regular and a map

$$\mathfrak{C}_\mathbb{A} \colon \bigcup_{a \in \mathcal{Z}} W(a) \to H_{\delta^*}$$

such that for each $b, b' \in \mathrm{dom}(\mathfrak{C}_\mathbb{A})$,

$$(\mathfrak{C}_\mathbb{A}(b) = \mathfrak{C}_\mathbb{A}(b') \Rightarrow \mathfrak{C}_\mathbb{Q}(\pi(b)) = \mathfrak{C}_\mathbb{Q}(\pi(b')) \wedge \pitchfork(b)(r) = \pitchfork(b')(r)),$$

for all $r \leq^* \pi(b), \pi(b')$.

Lemma 3.15. *If $(\pitchfork, \pi)$ is a c-forking projection and $\mathbb{Q}$ is λ-Knaster to $<\lambda$-Linked poset then so is $\mathbb{A}$.*

Proof. Fix $\mathcal{X} \in [\mathbb{A}]^\lambda$. Since $\mathbb{Q}$ is λ-Knaster to $<\lambda$-Linked there is a set $\mathcal{Y} \in [\pi``\mathcal{X}]^\lambda$ and $\mathfrak{C}_\mathbb{Q}$ as above. Hence, the definition of c-forking projection gives a compatibility map $\mathfrak{C}_\mathbb{A}$ for $\mathbb{A}$. Note that $\mathcal{Z} \in [\mathcal{X}]^\lambda$ and $\mathfrak{C}_\mathbb{A}$ witness that $\mathbb{A}$ is λ-Knaster to $<\lambda$-Linked (Definition 2.17). □

Fix $\alpha \in \mathcal{J}$. By virtue of our set-up assumptions, $\mathbb{Q} \restriction \alpha$ is λ-Knaster to $<\lambda$-Linked. Besides, there is a forking projection from $\mathbb{A} \restriction \alpha$ to $\mathbb{Q} \restriction \alpha$ (see Remark 3.12). Call it $(\pitchfork^\alpha, \pi^\alpha)$.

Lemma 3.16. *The pair $(\pitchfork, \pi)$ is a a c-forking projection from $\mathbb{A}$ to $\mathbb{Q}$.*
In particular, $\mathbb{A}$ is λ-Knaster to $<\lambda$-Linked.

Proof. Let $\mathcal{X} \in [\mathbb{A}]^\lambda$, $\mathcal{Y} \in [\mathbb{Q}]^\lambda$, and $\mathfrak{C}_\mathbb{Q} \colon \bigcup_{p \in \mathcal{Y}} W(p) \to H_\delta$ be as in Definition 3.14. Set $\delta^* := \max\{\delta, \kappa^+\}$ and define $\mathfrak{C}_\mathbb{A} \colon \bigcup_{a \in \mathcal{Z}} W(a) \to H_{\delta^*}$ as:

$$\mathfrak{C}_\mathbb{A}(q, \vec{Q}) := \langle \mathfrak{C}_\mathbb{Q}(q), \langle i, \alpha, \mathfrak{C}_\mathbb{Q}(r), Q_i(\alpha, \vec{\pi}, \cdot) \mid i \leq \gamma^{\vec{Q}}, \alpha \in \mathrm{dom}(Q_0), r \in W(q{\restriction}\alpha) \rangle \rangle,$$

where $\vec{\pi}$ is such that $r = q \restriction \alpha^\frown \vec{\pi}$ and $Q_i(\alpha, \vec{\pi}, \cdot)$ is the fiber map ranging over all suitable t's. Let $b = (q, \vec{Q}), b' = (q', \vec{Q}')$ be in $\mathrm{dom}(\mathfrak{C}_\mathbb{A})$ and suppose that $\mathfrak{C}_\mathbb{A}(q, \vec{Q}) = \mathfrak{C}_\mathbb{A}(q', \vec{Q}')$. Clearly, both q and q' have the same $\mathfrak{C}_\mathbb{Q}$-value.

Let $r \leq^* q, q'$. The argument that $\pitchfork(b)(r) = \pitchfork(b')(r)$ is the same as that from [PRS19, Lemma 6.13(8)] going over all possible coordinates $\alpha \in \mathrm{dom}(Q_0)$ within the Sharon strategies. □

Remark 3.17. The same result holds for the forking projection $(\pitchfork^\alpha, \pi^\alpha)$ for all $\alpha \in \mathcal{J}$. Thus, $\mathbb{A} \restriction \alpha$ is λ-Knaster to $<\lambda$-Linked.

3.3. Prikry property and killing non-reflecting stationary sets.

Lemma 3.18. *Forcing with $\mathbb{A}$ kills the stationarity of $\dot{T}^+$.*

Proof. Let $\mathbb{A}^\star \subseteq \mathbb{A}$ consisting of condition $(p, \vec{S})$ with $\lambda \in \mathrm{dom}(S_i)$ for all $i \leq \gamma^{\vec{S}}$. Clearly, $\mathbb{A}^\star$ is $\leq^*$-dense in $\mathbb{A}$. Let $\mathbb{A}^-$ denote the Sharon as defined in [PRS22, §4] with inputs $\mathbb{Q}$ and T^+. Arguing as in Lemma 3.7, $\mathbb{A}^\star$ projects onto $\mathbb{A}^-$ via $(p, \vec{S}) \mapsto (p, \langle S_i(\lambda) \mid i \leq \gamma^{\vec{S}}\rangle)$. In particular, $\mathbb{A}^\star$ kills the stationarity of $\dot{T}^+$ as so does $\mathbb{A}^-$ ([PRS22, Fact 4.10]). □

Recall the notion of a type over a pair $(\pitchfork, \pi)$ in [PRS22, Definitions 2.23].

Definition 3.19. Let $\mathrm{tp}\colon \mathbb{A} \to {}^{<\kappa^+}\omega$ be defined as follows:

$$\mathrm{tp}(p, \vec{S}) := \langle m(S_i) \mid i \leq \gamma^{\vec{S}}\rangle,$$

Also, define the maximal type, $\mathrm{mtp}(p, \vec{S}) = m(S_{\gamma^{\vec{S}}})$.

The arguments in [PRS22, Lemma 4.15] show that tp is a *type*. For the reader's convenience we reproduce the argument showing that the *ring poset*

$$\mathring{\mathbb{A}}_n := \{a \in \mathbb{A}_n \mid \pi(a) \in \mathring{\mathbb{Q}}_n \ \wedge\ \mathrm{mtp}(a) = 0\}$$

is dense in $\mathbb{A}_n$ (Clause (7) in [PRS22, Definition 2.23]).

Lemma 3.20. *The ring poset $\mathring{\mathbb{A}}_n$ is $\leq^*$-dense in $\mathbb{A}_n$.*

Proof. Let $(p, \vec{S}) \in \mathbb{A}_n$ be arbitrary and $\rho < \kappa^+$ be such that

$$\rho > \sup\{\max(S_i(\alpha, \cdot)) \mid \alpha \in \mathrm{dom}(S_0) \ \wedge\ \vec{\pi} \in \mathrm{dom}(S_i(\alpha)) \ \wedge\ i \leq \gamma^{\vec{S}}\}.$$

Note that this choice is possible because $|\mathrm{dom}(S_0)| \leq \kappa$. The argument in [PRS22, Claim 4.15.2] gives $q \leq^* p$ and ϱ with $(\varrho, q) \in R \restriction \lambda$.

Claim 3.20.1. *For each $\alpha \in \mathcal{J}$, if $(\varrho, q) \in R \restriction \lambda$ then $(\varrho, q \restriction \alpha) \in R \restriction \alpha$.*

Proof of claim. Assume $(\varrho, q) \in R \restriction \lambda$. Let $r \leq_{\mathbb{Q}\restriction\alpha} q \restriction \alpha$ be arbitrary and suppose towards a contradiction that $r \nVdash_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} \varrho \in \dot{C}_{\ell(r)}$. By $\leq_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}}$-extending r we may assume that $r \Vdash_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} \varrho \notin \dot{C}_{\ell(r)}$. Since $\restriction\alpha$ is a length-preserving projection we find a condition $q' \leq_{\mathbb{Q}} q$ such that $q' \restriction \alpha \leq_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} r$. Since $(\varrho, q) \in R \restriction \lambda$, $q' \Vdash_{\mathbb{Q}_{\ell(r)}} \varrho \in \dot{C}_{\ell(r)}$, which yields $q' \restriction \alpha \Vdash_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} \varrho \in \dot{C}_{\ell(r)}$ (because $\dot{C}_{\ell(r)}$ is a $(\mathbb{Q} \restriction \alpha)_{\ell(r)}$-name). This is a contradiction. □

Let $\vec{Q} := \langle Q_i \mid i \leq \gamma^{\vec{S}} + 1\rangle$ be with $\mathrm{dom}(Q_i) := \mathrm{dom}(S_0)$ and for each $\beta \in \mathrm{dom}(S_i)$, $Q_i(\beta)$ has the following set as a domain:

$$\textstyle\prod_{\ell(q) \leq n \leq k} \pi_{\mathrm{mc}(a_i^q)\,\mathrm{mc}(a_i^q \cap \beta)} \text{``} A_i^q.$$

Finally, $Q_i(\beta, \vec{\pi}, t)$ is defined by the following clauses

$$\begin{cases} S_i(\beta, \langle \pi_{\mathrm{mc}(a_i^q \cap \beta), \mathrm{mc}(a^{p_i} \cap \beta)}(\pi_j) \mid j < |\vec{\pi}|\rangle), & \text{if } i \leq \gamma^{\vec{S}}; \\ S_{\gamma^{\vec{S}}}(\beta, \langle \pi_{\mathrm{mc}(a_i^q \cap \beta), \mathrm{mc}(a^{p_i} \cap \beta)}(\pi_j) \mid j < |\vec{\pi}|\rangle) \cup \{\varrho\}, & \text{if } i = \gamma^{\vec{S}} + 1 \end{cases}$$

By the previous claim $(q, \vec{Q})$ is a well-defined condition in $\mathring{\mathbb{A}}_n$. Moreover, $(q, \vec{Q}) \leq^* (p, \vec{S})$. This accomplishes the proof of density of $\mathring{\mathbb{A}}_n^{\varsigma_n}$. □

Another important fact about the projection π is the following:

Fact 3.21. *$\mathring{\mathbb{A}}_n^\pi$ is κ^+-directed-closed.*

This fact can be proved along the lines of [PRS19, Lemma 6.15] using that the labeled p-trees have domains in $[\mathcal{J}]^{\leq\kappa}$.

Finally, $(\pitchfork, \pi)$ has the weak mixing property by the arguments in [PRS22, Lemma 4.16] at each of the various $\alpha \in \mathrm{dom}(S)$. Thus one obtains:

Lemma 3.22. tp *witnesses that $(\pitchfork, \pi)$ has the weak mixing property* (WMP)*. In particular, $\mathbb{A}$ has the Strong Prikry Property.* □

3.4. **Conclusion.** Putting everything into the same canopy we get:

Theorem 3.23. *Under the assumptions of Setup 3, there is a co-bounded (in λ) set $\mathcal{J} \subseteq \mathcal{I}$, $\lambda \in \mathcal{J}$, a sequence $\langle \mathbb{A} \restriction \alpha \mid \alpha \in \mathcal{J} \rangle$ of λ-Knaster to $<\lambda$-Linked weak-Σ-Prikry forcings with property $\mathcal{D}$ and projections $\langle \cdot \restriction \alpha \mid \alpha \in \mathcal{J} \rangle$ such that, for each $\alpha \in \mathcal{J}$, the following hold:*

1. *there is a forking projection from $\mathbb{A} \restriction \alpha$ to $\mathbb{M} \restriction \alpha$ with the WMP;*
2. *$\cdot \restriction \alpha$ is a length-preserving projection;*
3. *for each $n < \omega$, $(\mathring{\mathbb{A}} \restriction \alpha)_n^\pi$ is κ^+-directed-closed;*
4. *$\mathbb{1} \Vdash_{\mathbb{A} \restriction \alpha} \mu = \kappa^+$;*
5. *$\mathbb{A} \restriction \alpha$ is a subset of H_λ;*
6. *$\mathbb{1} \Vdash_{\mathbb{A} \restriction \alpha}$ "$\dot{T}^+$ is non-stationary".*

4. The main iteration

In this section we define a weak-Σ-Prikry forcing yielding the configuration of our main theorems. The poset will be a Σ-Prikry-styled λ-length iteration $\mathbb{P}_\lambda$, with support $\leq \kappa$, starting with the poset $\mathbb{M}$ of §2.1. The iteration will kill all the non-reflecting stationary subsets of κ^+ invoking the revised Sharon's functor $\mathbb{A}(\cdot, \cdot)$ of p. 17.

4.1. **The iteration and its projections.** Let $\psi \colon \lambda \to H_\lambda$ be a surjection, such that for each inaccessible $\beta \leq \lambda$, $\psi \restriction \beta$ is a bookkeeping function; namely, $\mathrm{Im}(\psi) = H_\beta$ and the ψ-preimage of each $x \in H_\beta$ is cofinal in β.

We define a λ-length iteration $\mathbb{P}_\lambda$ with support $\leq \kappa$ as follows. The first step is the poset $\mathbb{P}_1 := \mathbb{M}$. The successor step is given by the functor $\mathbb{A}(\cdot, \cdot)$ of p. 17, using the bookkeeping function ψ.

Similarly, for each inaccessible $\beta \leq \lambda$, $\mathbb{Q}^\beta$ is defined to be the β-length, $\leq \kappa$ support iteration, where the first step is $\mathbb{P}_1 \restriction \beta$ and the successor step is given by the function from the previous section, using $\psi \restriction \beta$ as a bookkeeping function. Note that $\mathbb{P}_\lambda$ is simply $\mathbb{Q}^\lambda$.

Notation 4.1. For $\alpha \in [1, \beta)$ denote $\mathbb{Q}^\beta_\alpha$ the α^{th}-stage of the iteration $\mathbb{Q}^\beta$. Note that $\mathbb{Q}^\lambda_\alpha = \mathbb{P}_\alpha$ (the α-th stage of the iteration $\mathbb{P}_\lambda$).

In a mild abuse of notation if $p \in \mathbb{Q}^\beta_\alpha$ and $\bar{\alpha} < \alpha$, $p \restriction \bar{\alpha}$ denotes the natural restriction of p to a condition in $\mathbb{Q}^\beta_{\bar{\alpha}}$, the $\bar{\alpha}^{\text{th}}$-stage subiteration of $\mathbb{Q}^\beta_\alpha$.

Lemma 4.2. *For each $0 < \alpha < \lambda$ and for all large enough inaccessible $\beta < \lambda$, there are length-preserving projections $\rho^\beta_\alpha \colon \mathbb{P}_\alpha \to \mathbb{Q}^\beta_\alpha$. Moreover, these projections are truncation-stable in the following sense:*

$$(\rho) \qquad \forall \bar{\alpha} < \alpha \; (\rho^\beta_\alpha(p) \restriction \bar{\alpha} = \rho^\beta_{\bar{\alpha}}(p \restriction \bar{\alpha})).$$

Proof. Let us argue by induction on α. The base case $\alpha = 1$ holds for all inaccessible $\beta < \lambda$, since $\mathbb{P}_1$ projects to $\mathbb{P}_1 \restriction \beta$ by our findings in the previous sections. Suppose that the result holds for all $\bar{\alpha} < \alpha$.

► $\alpha = \bar{\alpha} + 1$. By induction, for all large β, we have a projection $\rho : \mathbb{P}_{\bar{\alpha}} \to \mathbb{Q}^\beta_{\bar{\alpha}}$. By Theorem 3.23, for all large β, this induces a projection $\mathbb{A}(\mathbb{P}_{\bar{\alpha}}, .) \to \mathbb{A}(\mathbb{Q}^\beta_{\bar{\alpha}}, .)$. Since $\mathbb{A}(\mathbb{P}_{\bar{\alpha}}, .) = \mathbb{P}_\alpha$ and $\mathbb{A}(\mathbb{Q}^\beta_{\bar{\alpha}}, .) = \mathbb{Q}^\beta_\alpha$, we are done.

► $\alpha < \lambda$ is limit. Then for all $\bar{\alpha} < \alpha$, for all large β, we have projections $\mathbb{P}_{\bar{\alpha}} \to \mathbb{Q}^\beta_{\bar{\alpha}}$. For all such β, define $\rho^\beta_\alpha \colon \mathbb{P}_\alpha \to \mathbb{Q}^\beta_\alpha$ by setting

$$\rho^\beta_\alpha(p) := \bigcup_{\bar{\alpha} < \alpha} \rho^\beta_{\bar{\alpha}}(p \restriction \bar{\alpha}).$$

Using the induction hypothesis one can check that ρ^β_α satisfies (ρ). Thus, $\rho^\beta_\alpha(p)$ is a well-defined condition in $\mathbb{Q}^\beta_\alpha$ for all $p \in \mathbb{P}_\alpha$. It is routine (if a little technical) to check that this is a projection. □

Lemma 4.3. *Suppose that $j \colon V \to M$ is an elementary embedding with $\mathrm{crit}(j) = \lambda$. Then, $j(\mathbb{P}_\lambda)$ projects to $\mathbb{P}_\lambda$.*

Proof. For simplicity, let us write $\mathbb{P}$ for $\mathbb{P}_\lambda$. It suffices to show that $j(\mathbb{P})_\lambda$ projects to $\mathbb{P}$ because $j(\mathbb{P})$ projects to $j(\mathbb{P})_\lambda$ via the map $p \mapsto p \restriction \lambda$.

By Lemma 4.2, $\{\beta < \lambda \mid \mathbb{P}_\alpha \text{ projects to } \mathbb{Q}^\beta_\alpha\}$ is $\mathcal{U}$-large for all $\alpha \in [1, \lambda)$, where $\mathcal{U}$ is the normal measure obtained from j. In particular, for each such $\alpha < \lambda$, $j(\mathbb{P})_\alpha$ projects to $j(\mathbb{Q})^\lambda_\alpha = \mathbb{P}_\alpha$ via a map ϑ^λ_α witnessing (ρ) of Lemma 4.2.

Now, for each $p \in j(\mathbb{P})_\lambda$ let $\alpha(p) < \lambda$ be the first ordinal above the support of p. This ordinal exists as $j(\mathbb{P})$ is an iteration with support of size $\leq j(\kappa) = \kappa$. Next, define $\vartheta(p) := \vartheta^\lambda_{\alpha(p)}(p)$. Using stability under truncation (i.e., equation (ρ)) it is routine to check that ϑ defines a projection. □

4.2. **Chain condition.** In this section we show that our main iteration $\mathbb{P}_\lambda$ is λ-Knaster. For this we should first verify that all the intermediate stages $\mathbb{P}_\alpha$ are λ-Knaster to $<\lambda$-Linked (see Definition 2.17).

Lemma 4.4. *For $1 \leq \alpha < \beta < \lambda$, $(\pitchfork_{\beta,\alpha}, \pi_{\beta,\alpha})$ is a c-forking projection.*

In particular, $\mathbb{P}_\alpha$ is λ-Knaster to $<\lambda$-Linked for all $1 \leq \alpha < \lambda$.

Proof. The second claim will follow from Lemma 2.21 and Remark 3.15 once we establish that $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ is a c-forking projection (Definition 3.14).

We proceed by induction on $1 < \beta < \lambda$. By the previous section, the pair $(\pitchfork_{2,1}, \pi_{2,1})$ is a c-forking projection. So, suppose that $\langle(\pitchfork_{\beta,\alpha}, \pi_{\beta,\alpha}) \mid 1 \leq \alpha < \beta < \gamma\rangle$ is a sequence of c-forking projections.

Case $\gamma = \bar{\gamma} + 1$: This case follows by the same arguments as in Lemma 3.16.

Case $\gamma \in \mathrm{acc}(\lambda)$: Fix $\beta \in [1, \gamma)$ and let us show that $(\pitchfork_{\gamma,\beta}, \pi_{\gamma,\beta})$ is a c-forking projection. Fix $\mathcal{X} \in [\mathbb{P}_\gamma]^\lambda$, $\mathcal{Y} \in [\mathbb{P}_\beta]^\lambda$ and $\mathfrak{C}_\beta \colon \bigcup_{p \in \mathcal{Y}} W(p) \to H_\delta$. For each $\epsilon \in [\beta, \gamma)$ set $\mathcal{X}_\epsilon := \pi_{\gamma,\epsilon}``\mathcal{X}$. By our induction hypothesis, $(\pitchfork_{\epsilon,\beta}, \pi_{\epsilon,\beta})$ is a c-forking projection. Invoke this property with respect to $(\mathcal{X}_\epsilon, \mathcal{Y}, \mathfrak{C}_\beta)$ and obtain a map $\mathfrak{C}_\epsilon \colon \bigcup_{p \in \mathcal{Z}_\epsilon} W(p) \to H_{\delta_\epsilon}$, $\delta_\epsilon \geq \epsilon$, witnessing the statement in Definition 3.14. Recall that $\mathcal{Z}_\epsilon = \{p \in \mathcal{X}_\epsilon \mid \pi_{\epsilon,\beta}(p) \in \mathcal{Y}\}$.

Set $\mathcal{Z}_\gamma := \{p \in \mathcal{X} \mid \pi_{\gamma,\beta}(p) \in \mathcal{Y}\}$ and let $\delta \leq \delta^* < \lambda$ be a regular cardinal above $(\sup_{\beta<\epsilon<\gamma} \delta_\epsilon)^+$. Define $\mathfrak{C}_\gamma \colon \bigcup_{p \in \mathcal{Z}_\gamma} W(p) \to H_{\delta^*}$ as

$$\mathfrak{C}_\gamma(q) := \{\langle \epsilon, \mathfrak{C}_\epsilon(q \restriction \epsilon)\rangle \mid \epsilon \in [\beta, \gamma)\}.$$

Note that $\mathfrak{C}_\gamma$ is well-defined: First, each $q \in W(p)$ with $p \in \mathcal{Z}_\gamma$ satisfies that $q \restriction \epsilon \in W(p \restriction \epsilon)$ and $p \restriction \epsilon \in \mathcal{Z}_\epsilon$ – thus, $\mathfrak{C}_\epsilon(q \restriction \epsilon)$ makes sense. Second, our choice of δ^* ensures that the range of $\mathfrak{C}_\gamma(q)$ is included in H_{δ^*}.

Let us show that the map $\mathfrak{C}_\gamma$ witnesses that $(\pitchfork_{\gamma,\beta}, \pi_{\gamma,\beta})$ is a c-forking projection. Let $q, q' \in \mathrm{dom}(\mathfrak{C}_\gamma)$ and suppose that $\mathfrak{C}_\gamma(q) = \mathfrak{C}_\gamma(q')$.

- ▶ Clearly, $\mathfrak{C}_\beta(q \restriction \beta) = \mathfrak{C}_\beta(q' \restriction \beta)$.
- ▶ Fix $r \leq^*_\beta q \restriction \beta, q' \restriction \beta$. By definition of $\pitchfork_{\gamma,\beta}$ for limit ordinals γ,

$$\pitchfork_{\gamma,\beta}(q)(r) = \bigcup_{\beta \leq \epsilon < \gamma} \pitchfork_{\epsilon,\beta}(q \restriction \beta)(r).$$

Since for each $\epsilon \in (\beta, \gamma)$, $\mathfrak{C}_\epsilon(q \restriction \epsilon) = \mathfrak{C}_\epsilon(q' \restriction \epsilon)$ and $\mathfrak{C}_\epsilon$ witnessed that $(\pitchfork_{\epsilon,\beta}, \pi_{\epsilon,\beta})$ is a c-forking projection, $\pitchfork_{\epsilon,\beta}(q \restriction \beta)(r) = \pitchfork_{\epsilon,\beta}(q' \restriction \beta)(r)$. Since this equality holds true for every $\epsilon \in [\beta, \gamma)$ we conclude that $\pitchfork_{\gamma,\beta}(q)(r) = \pitchfork_{\gamma,\beta}(q')(r)$. □

Lemma 4.5. *$\mathbb{P}_\lambda$ is λ-Knaster.*

Proof. Let $X \in [\mathbb{P}_\lambda]^\lambda$ and set $\mathcal{S} := \{\mathrm{cl}(B_p) \mid p \in X\}$. Here we use the notation of [PRS22, §3.1] where we stipulated $B_p := \{\gamma + 1 < \lambda \mid p(\gamma) \neq \emptyset\}$. Similarly, by $\mathrm{cl}(B_p)$ we denote the ordinal closure of B_p.

Since $\mathcal{S} \subseteq [\lambda]^{\leq\kappa}$ and λ is an inaccessible cardinal one can shrink X to $Y \in [X]^\lambda$ so that $\{\mathrm{cl}(B_p) \mid p \in Y\}$ forms a Δ-system. Say this Δ-system has root Δ and set $\chi := \sup(\Delta)$, which clearly is smaller than λ.

By Lemma 4.4, $\mathbb{P}_\chi$ is λ-Knaster to $<\lambda$-Linked. In particular, there is a set $Y_\chi \in [\pi_{\lambda,\chi}``Y]^\lambda$ where every two conditions are $\leq^*_\chi$-compatible. It is not hard to verify that $Z := \{p \in Y \mid \pi_{\lambda,\chi}(p) \in Y_\chi\}$ is a collection of pairwise compatible conditions. The basic idea is to go over the supports of the conditions in Z and take pitchforks finding $\leq^*$-lower bounds along the way. For details, see the argument in [PRS22, Claim 3.14.1]. □

5. The Ineffable Tree Property at double successors

In this section we prove that the *Ineffable Tree Property* (ITP) holds after forcing with the Σ-Prikry-styled iteration of §4.

Definition 5.1. A sequence $\langle d_x \mid x \in \mathcal{P}_\lambda(\theta)\rangle$ is a *thin* $\mathcal{P}_\lambda(\theta)$*-list* if $d_x \subseteq x$ and $|\{d_x \cap c \mid c \subseteq x\}| < \lambda$ for club many $c \in \mathcal{P}_\lambda(\theta)$.[12]

Given a thin $\mathcal{P}_\lambda(\theta)$-list $d = \langle d_x \mid x \in \mathcal{P}_\lambda(\theta)\rangle$ and $b \subseteq \theta$ one says that:

- b is a *cofinal branch* through d if for all $x \in \mathcal{P}_\lambda(\theta)$, there is $y \in \mathcal{P}_\lambda(\theta)$ with $x \subset y$, such that $b \cap x = d_y \cap x$.
- b is an *ineffable branch* through d if $\{x \in \mathcal{P}_\lambda(\theta) \mid d_x = b \cap x\}$ is stationary in $\mathcal{P}_\lambda(\theta)$.

The *Ineffable Tree Property* holds at λ (in symbols, $\mathrm{ITP}(\lambda)$) if for all regular cardinal $\theta > \lambda$ every thin $\mathcal{P}_\lambda(\theta)$-list d carries an ineffable branch through it.

Suppose that λ is supercompact. Fix a regular cardinal $\theta > \lambda$ and let $j\colon V \to M$ be an embedding witnessing θ-supercompactness of λ. By $\mathbb{P}_\lambda$ we will denote the Σ-Prikry-styled iteration as described in §4. Namely, the first step of the iteration is the Mitchell EBPF of Definition 2.6 and the other stages are constructed by invoking the functor of §3.

Let $G \subseteq \mathbb{P}_\lambda$ a generic filter over V and denote by π the projection from $j(\mathbb{P}_\lambda)$ to $\mathbb{P}_\lambda$ (see Lemma 4.3). Let $H \subseteq j(\mathbb{P}_\lambda)/G$ generic over $V[G]$. Since $\mathbb{P}_\lambda \subseteq H_\lambda$, $j``G \subseteq G * H$, and so j lifts in $V[G*H]$ to $j\colon V[G] \to M[G*H]$.

Suppose that $d = \langle d_x \mid x \in \mathcal{P}_\lambda(\theta)^{V[G]}\rangle$ is a thin $\mathcal{P}_\lambda(\theta)$-list in $V[G]$. By standard arguments, $b := \{\alpha < \theta \mid j(\alpha) \in j(d)_{j``\theta}\}$ is a $V[G]$-ineffable branch through d (see e.g. [HS19, p.5]). Also, $b \in V[G*H]$ and it is $<\lambda$-approximated in $V[G]$; namely, $b \cap x \in V[G]$ for all $x \in \mathcal{P}_\lambda(\theta)^{V[G]}$.

Working over V, let $\dot{b}$ be a $j(\mathbb{P}_\lambda)$-name for the branch b such that

$$\mathbb{1} \Vdash_{j(\mathbb{P}_\lambda)} \text{“}\dot{b} \text{ is ineffable } \wedge \forall x \in \mathcal{P}_\lambda(\theta)^{V[\dot{G}]}(\dot{b} \cap x \in V[\dot{G}])\text{”},$$

where $\dot{G}$ is the standard name for the generic filter of $\mathbb{P}_\lambda$.

For the rest of this section we suppose towards a contradiction that $\dot{b}$ is not forced (by $\mathbb{1}$) to be in $V[\dot{G}]$. By λ-ccness of $\mathbb{P}_\lambda$, $\mathcal{P}_\lambda(\theta)^V$ is $\subseteq$-unbounded in $\mathcal{P}_\lambda(\theta)^{V[G]}$, hence we shall be working with x's in the ground model V.

In Definition 2.3 we introduced the fusion ordering $\leq^{*,k}$ of the EBPF forcing. The $\leq^{*,k}$-order lifts naturally to $j(\mathbb{P}_\lambda)$ as follows:

Definition 5.2 (Fusion ordering)**.** For each $k < \omega$ and $u, v \in j(\mathbb{P}_\lambda)$,

$$u \leq^{*,k} v :\Longleftrightarrow u \leq^* v \text{ and } \mathsf{EBPF}(u) \leq^{*,k} \mathsf{EBPF}(v),$$

where $\mathsf{EBPF}(u)$ and $\mathsf{EBPF}(v)$ denote the EBPF-part of u and v. The ordering $\leq^{*,k,-}$ is defined analogously just requiring $\mathsf{EBPF}(u) \leq^{*,k,-} \mathsf{EBPF}(v)$.

Recall that $\mathring{\mathbb{P}}_\lambda$ denotes the *ring poset* defined in [PRS23, Definition 7.6]. For each $n < \omega$, this poset has the following two key properties:

(1) $(\mathring{\mathbb{P}}_\lambda)_n$ is a dense $\aleph_1$-directed-closed subforcing of $(\mathbb{P}_\lambda)_n$
(2) $(\mathring{\mathbb{P}}_\lambda)_n^{\pi_{\lambda,1}} := ((\mathring{\mathbb{P}}_\lambda)_n, \leq^{\pi_{\lambda,1}})$ is κ^+-directed closed.

[12] According to Jech [Jec73], $\mathcal{C} \subseteq \mathcal{P}_\lambda(\theta)$ is a club if it is closed and unbounded in the following sense: **Closed**: Given a $\subseteq$-increasing sequence $\langle c_\alpha \mid \alpha < \beta\rangle \subseteq \mathcal{C}$ with $\beta < \lambda$, $\bigcup_{\alpha<\beta} c_\beta \in \mathcal{C}$; **Unbounded:** For each $x \in \mathcal{P}_\lambda(\theta)$ there is $c \in \mathcal{C}$ such that $x \subseteq c$.

Lemma 5.3. *There is $\bar{n} < \omega$ and $u \in j(\mathbb{P}_\lambda)$ such that for all $\ell(u) \leq k < \omega$ and $v \leq^* u$ there is $x \in \mathcal{P}_\lambda(\theta)$ such that for all $y \in \mathcal{P}_\lambda(\theta)$ with $x \subseteq y$ there is $w \leq^{*,k,-} v$ in $j(\mathring{\mathbb{P}}_\lambda)$ all of whose $\bar{n}$-extensions decide the value of $\dot{d} \cap y$.*

Proof. The proof is the same as in [HS19, Lemma 4.4] using the following strengthening of the Strong Prikry Property of $j(\mathbb{P}_\lambda)$: Let $u \in j(\mathbb{P}_\lambda)$, $D \subseteq j(\mathbb{P}_\lambda)$ be dense open and $k \geq \ell(u)$. Then, there is $v \leq^{*,k,-} u$ and $\bar{n} < \omega$ such that every $\bar{n}$-extension of v enters D. This strengthening of the SPP is established as follows: First, it holds for $j(\mathbb{M})$ because it holds for the EBPF. Second, the proof of property $\mathcal{D}$ given in [PRS22, Lemma 3.11] shows that a witness for the SPP for $j(\mathbb{P}_\lambda)$ has as a first coordinate a condition witnessing the same fact for $j(\mathbb{P}_1)$. Since the relation $\leq^{*,k,-}$ just depends on what occurs at the first coordinate we are done. □

Definition 5.4. Let $u, v \in j(\mathbb{P}_\lambda)$ be with the same length ℓ and $k \geq \ell$.

Write $u^{\frown +k} \leq v^{\frown +k}$ if the first $(k-\ell)$-many (a, A)-parts of $\mathsf{EBPF}(u)$ and $\mathsf{EBPF}(v)$ are the same and for each $\vec{\nu} \in \prod_{\ell \leq i \leq k} A_i$, $u^\frown \vec{\nu} \leq^* v^\frown \vec{\nu}$.

Remark 5.5. If $u^{\frown +k} \leq v^{\frown +k}$ then $\mathsf{EBPF}(u) \leq^* \mathsf{EBPF}(v)$. However, this may not be the case for the Mitchell/Sharon-parts of u and v: Let c^u and c^v denote the Mitchell-parts of u and v. From $u^{\frown +k} \leq v^{\frown +k}$ it is easy to show that $\mathrm{dom}(c^u) \supseteq \mathrm{dom}(c^v)$. However, it might well be that

$$c^u(\alpha, \vec{\nu} \restriction \alpha) \nleq c^v(\alpha, \vec{\nu} \restriction \alpha)$$

for $\alpha \in \mathrm{dom}(c^v)$ and a short $\vec{\nu} \in \prod A_i$ (i.e, $|\vec{\nu}| < (k-\ell)$). Here $\vec{\nu} \restriction \alpha$ denotes the sequence given by the successive αth-projections of $\vec{\nu}$.

Morally speaking, $u^{\frown +k} \leq v^{\frown +k}$ means that $u \leq v$ holds, modulo some finite error. Nonetheless, forcing-wise u behaves as a strengthening of v, in the sense that, if $u^{\frown +k} \leq v^{\frown +k}$ then $u \Vdash v \in \dot{G}$. Thus, for any sentence φ in the language of forcing of $j(\mathbb{P}_\lambda)$ if $v \Vdash \varphi$ then $u \Vdash \varphi$. □

Lemma 5.6.

(1) $\langle j(\mathring{\mathbb{P}}_\lambda), {}^{\frown +k} \rangle$ is κ_{k+1}-closed.

(2) If $u^{\frown +k} \leq v^{\frown +k}$ then there is $w \leq^{*,k} v$ such that $w^{\frown +k} \leq u^{\frown +k}$.

Proof. (1) Let $\langle u_\alpha \mid \alpha < \theta < \kappa_{k+1} \rangle$ be a ${}^{\frown +k}$-decreasing sequence in $j(\mathring{\mathbb{P}}_\lambda)$. Let $\vec{\nu} \in \prod_{\ell \leq i \leq k} A_i$ be a sequence in the common measure one sets. By definition, for each $\vec{\nu}$, $\langle u_\alpha{}^\frown \vec{\nu} \mid \alpha < \theta \rangle$ is $\leq^*$-decreasing in $j(\mathring{\mathbb{P}}_\lambda)$. Using the closure of the $\leq^*$-ordering for the MEBPF define a lower bound for the sequence of first coordinates (i.e. the MEBPF parts) of $\langle u_\alpha{}^\frown \vec{\nu} \mid \alpha < \theta \rangle$. Next, replace the first coordinate of $u_\alpha{}^\frown \vec{\nu}$ by this lower bound and use the closure of the ring poset $j(\mathring{\mathbb{P}}_\lambda)$ under κ^+-sequences with the same first coordinate. Let $u_{\vec{\nu}}$ be the resulting bound. Inductively, arrange that $\langle u_{\vec{\nu}} \mid \vec{\nu} \in \prod A_i \rangle$ are diagonalizable; namely, arrange the $u_{\vec{\nu}}$'s so that there is $u_\theta \leq^{*,k} u_0$ such that $u_\theta{}^\frown \vec{\nu} \leq^* u_{\vec{\nu}}$ for all $\vec{\nu}$. Clearly $u_\theta{}^{\frown +k} \leq u_\alpha{}^{\frown +k}$.

(2) Let us describe how to define such a w. The EBPF-part of w, p_w, is the same as that of u. The Mitchell-part of w is c_w where: $\mathrm{dom}(c_w) = \mathrm{dom}(c_u)$,

for each $\alpha \in \mathrm{dom}(c_w)$ and $\vec{\pi} \in \mathrm{dom}(c_w(\alpha))$, $c_w(\alpha, \vec{\pi}) = c_u(\alpha, \vec{\pi})$ in case $\alpha \notin \mathrm{dom}(c_v)$; otherwise, $c_w(\alpha, \vec{\pi})$ is defined according to the following cases:

$$c_w(\alpha, \vec{\pi}) := \begin{cases} c_u(\alpha, \vec{\pi}), & \text{if } |\vec{\pi}| \geq k - \ell; \\ c_v(\alpha, \langle \pi_{\mathrm{mc}(a_i^u \cap \alpha), \mathrm{mc}(a_i^v \cap \alpha)}(\pi_i) \mid i \leq |\vec{\pi}| \rangle), & \text{otherwise.} \end{cases}$$

The Sharon-part of w is defined by going over the support of the stronger condition u. Let us just describe the case relative to the first coordinate. Let $\vec{S} = \langle S_i \mid i \leq \gamma^{\vec{S}} \rangle$ and $\vec{Q} = \langle Q_i \mid i \leq \gamma^{\vec{Q}} \rangle$ be the first Sharon strategies of u and v, respectively. The Sharon strategy of w is $\vec{R} := \langle R_i \mid i \leq \gamma^{\vec{S}} \rangle$: For each $i \leq \gamma^{\vec{S}}$, $\mathrm{dom}(R_i) = \mathrm{dom}(S_i)$ and for each $\alpha \in \mathrm{dom}(R_i)$ and $\vec{\pi} \in \mathrm{dom}(R_i)$ define $Q_i(\alpha, \vec{\pi})$ as: $R_i(\alpha, \vec{\pi}) := S_i(\alpha, \vec{\pi})$, provided $\alpha \notin \mathrm{dom}(Q_i)$; otherwise, define $R_i(\alpha, \vec{\pi})$ as before replacing c_u (resp. c_v) by S_i (resp. Q_i) whenever $i \leq \gamma^{\vec{Q}}$, or simply using S_i if $i > \gamma^{\vec{Q}}$.

We leave the details that w is as desired to the interested reader. □

Remark 5.7. The previous proof is flexible enough to yield the following variant of Clause (2) above: Suppose that $u \leq^* v$ and $u^\frown \vec{\nu} \leq^* v^\frown \vec{\nu}$ for all $\vec{\nu} \in \prod_{i=\ell(u)}^{k} A_i^u$, but the first $(k-\ell)$-measure one sets of u and v are not the same (so, possibly $u^{\frown +k} \not\leq v^{\frown +k}$).[13] Then, there is $w \leq^{*,k} v$ such that

$$w^\frown \vec{\nu} = u^\frown \vec{\nu} \text{ for all } \vec{\nu} \in \textstyle\prod_{i=\ell(u)}^{k} A_i^u.$$

The next is the key technical lemma. Fix $\bar{n}$ as in Lemma 5.3, and work below the condition given by that lemma.

Lemma 5.8 (Splitting Lemma)**.** *Let $u \in j(\mathbb{P}_\lambda)$, $k \geq \ell(u) + \bar{n}$ and δ be a regular cardinal such that $2^{\kappa_k} < \delta < \kappa_{k+1}$. Then, there is a sequence of conditions $\langle u_\xi \mid \xi < \delta \rangle$, a set $y \in \mathcal{P}_\lambda(\theta)$ and $\bar{u} \leq^{*,k,-} \pi(u)$ such that:*

1. *$u_\xi \leq^{*,k,-} u$ (i.e, $u_\xi \leq^* u$ and $a_i^{u_\xi} = a_i^u$ for all $\ell(u) \leq i \leq k$);*
2. *every $\bar{n}$-extension of u_ξ decides a value for $\dot{b} \cap y$;*
3. *for all $\ell(u) \leq i \leq k$, $A_i^{u_\xi} = A_i$ for some constant A_i;*
4. *$\bar{u}^{\frown +k} \leq \pi(u_\xi)^{\frown +k}$ for all $\xi < \delta$;*
5. *let $\xi \neq \zeta$ and suppose that v and w are $\bar{n}$-extensions of u_ξ and u_ζ, respectively. Then, $\bar{u}$ forces (in the poset $\mathbb{P}_\lambda$) that the values of $\dot{b} \cap y$ decided by v and w are different.*

Proof. In the first stage of the proof we construct a sequence of conditions $\langle (u_\xi, v_\xi) \mid \xi < \delta \rangle$ in $j(\mathbb{P}_\lambda)$ and of elementary submodels $\langle \mathcal{M}_\xi \mid 0 < \xi < \delta \rangle$ as follows. Set $u_0 = v_0 := u$ and let $\mathcal{M}_1 \prec H_\chi$ be with $|\mathcal{M}_1| < \delta$, $u_0 \in \mathcal{M}_1$ and $\mathcal{M}_1^{\kappa_k} \subseteq \mathcal{M}_1$. Suppose that $\langle (u_\xi, v_\xi) \mid \xi < \zeta \rangle$ and $\langle \mathcal{M}_\xi \mid 0 < \xi < \zeta \rangle$ have been defined in a way that the v_ξ's are $\leq^{*,k}$-decreasing (i.e., they have the same (a, A)-part as u up to and including k) and

- $u_{\xi+1} \leq^{*,k,-} v_\xi$ for all ξ.
- u_ξ is in the ring poset $j(\mathring{\mathbb{P}}_\lambda)$ and the delay of v_ξ is $\leq k - \ell$.

[13] Recall that this latter was required in the definition of $^{\frown +k}$ given in Definition 5.4.

If ζ is a limit ordinal set $\mathcal{M}_\zeta := \bigcup_{\xi<\zeta}\mathcal{M}_\xi$; otherwise, $\mathcal{M}_\zeta$ is such that

$$u_{\zeta-1}\in\mathcal{M}_\zeta,\ |\mathcal{M}_\zeta|<\delta,\ \mathcal{M}_{\zeta-1}\subseteq\mathcal{M}_\zeta \text{ and } \mathcal{M}_\zeta^{\kappa_k}\subseteq\mathcal{M}_\zeta.$$

Let v be a $\leq^{*,k}$-lower bound for the previous v_ξ. This choice is possible: first, we have enough closure to take a $\leq^{*,k}$-lower bound for the Mitchell-part of v; second, we can modify the v_ξ's to have Mitchell-part this $\leq^{*,k}$-lower bound and use the κ^+-closure of the ring $j(\mathring{\mathbb{P}}_\lambda)$ with respect to sequences of conditions with common Mitchell-part. Moreover, v has delay $k-\ell$.[14]

Let $u_\zeta \leq^{*,k,-} v$ be in $j(\mathring{\mathbb{P}}_\lambda)$ all of whose $\bar{n}$-extensions decide $\dot{b}\cap y_\zeta$ where y_ζ is the θ-trace of the model $\mathcal{M}_\zeta$; namely, $y_\zeta := \mathcal{M}_\zeta\cap\theta$. (To choose u_ζ we use Lemma 5.3). Denote the EBPF/Mitchell-parts of u_ζ by p_ζ and c_ζ, respectively. The EBPF/Mitchell part of v are q and d. Note that $\mathrm{dom}(d)\subseteq\mathrm{dom}(c_\zeta)$. Next, we define v_ζ. Informally, v_ζ is the "amalgamation" of v and u_ζ i.e. it is below v and a certain restriction of it is below u_ζ.

First, let us define the EBPF-part of v_ζ:

$$q_\zeta := \langle f_0^\zeta,\dots,f_{\ell-1}^\zeta,(a_\ell,A_\ell,f_\ell^\zeta),\dots,(a_k,A_k,f_k^\zeta),(p_\zeta)_{k+1},(p_\zeta)_{k+2},\cdots\rangle$$

where f_i^ζ's are the Cohens of u_ζ, the (a_i,A_i)'s are from the initial condition u and $(p_\zeta)_i$ is the ith-entry of the EBPF-part of u_ζ.

The Mitchell-part of v_ζ, denoted d_ζ, is defined as follows: Let d_ζ be the function with $\mathrm{dom}(d_\zeta):=\mathrm{dom}(c_\zeta)$ and for each $\alpha\in\mathrm{dom}(d_\zeta)$,

$$\mathrm{dom}(d_\zeta(\alpha)) := [\textstyle\prod_{\ell\leq i\leq t}\pi_{\mathrm{mc}(a_i^{q_\zeta}),\mathrm{mc}(a_i^{q_\zeta}\cap\alpha)}\text{``}A_i^{q_\zeta}]^{<\omega}.$$

Let us say that $\vec{\pi}\in\mathrm{dom}(d_\zeta(\alpha))$ is *good* if $|\vec{\pi}|\geq k-\ell$ and $\vec{\pi}\in\mathrm{dom}(c_\zeta(\alpha))$.

If $\vec{\pi}$ is good, then for all $\alpha\in\mathrm{dom}(d_\zeta)$, define

$$d_\zeta(\alpha,\vec{\pi}) = c_\zeta(\alpha,\vec{\pi})$$

Otherwise, if $\vec{\pi}$ is not good, we split the definition into two cases:

$$d_\zeta(\alpha,\vec{\pi}) := \begin{cases} d(\alpha,\langle\pi_{\mathrm{mc}(a_i^{q_\zeta}\cap\alpha),\mathrm{mc}(a_i^{q}\cap\alpha)}(\pi_i)\mid \ell\leq i\leq t\rangle), & \text{if } \alpha\in\mathrm{dom}(d);\\ \{\langle\check{\varnothing},\mathbb{1}_{\mathbb{P}\restriction\alpha}\rangle\}, & \text{otherwise.}\end{cases}$$

Note that the a-part of q_ζ and q are the same up to k so the above projection in the first $(k-\ell)$-coordinates equals $\pi_{\mathrm{mc}(a_i^{q_\zeta}),\mathrm{mc}(a_i^{q}\cap\alpha)}(\pi_i)=\pi_i$.

It is not hard to check that (q_ζ,d_ζ) is a condition in $j(\mathbb{M})$ - the key point being $(q_\zeta\restriction\alpha)^\frown\vec{\pi}=(p_\zeta\restriction\alpha)^\frown\vec{\pi}$ for all good $\vec{\pi}$. By the construction it is also the case that (q_ζ,d_ζ) is $\leq^{*,k}$-stronger than the Mitchell-part of v. The pair (q_ζ,d_ζ) will be the eventual $j(\mathbb{M})$-part of the future condition v_ζ.

Let us define the Sharon-part of v_ζ. For each $\vec{\nu}\in\prod_{\ell\leq i\leq k}A_i$ define

$$u_{\vec{\nu}} := \begin{cases} u_\zeta{}^\frown\vec{\nu}, & \text{if } \vec{\nu}\in\prod_{\ell\leq i\leq k}A_i^{q_\zeta};\\ v^\frown\vec{\nu}, & \text{otherwise.}\end{cases}$$

[14]Even though the v_ξ's do not have null delay these are uniformly bounded by a fixed integer. This is enough to be able to take lower bounds and get the same delay.

Clearly, $u_{\vec{\nu}} \leq^* v^\frown\vec{\nu}$ and the delay of the $u_{\vec{\nu}}$ is exactly 0. For the Sharon-part of u_ζ we take the diagonalization of the above $u_{\vec{\nu}}$'s. More precisely, by the *Weak Mixing lemma* ([PRS22, Lemma 3.10]) there is a condition $v_\zeta \leq^{*,k} v$ with $v_\zeta \restriction 1 = (q_\zeta, d_\zeta)$, the delay of v_ζ is $k-\ell$ and $v_\zeta{}^\frown\vec{\nu} \leq^* u_{\vec{\nu}}$. This completes the definition of v_ζ and the inductive construction.

After this process we get $\langle (u_\xi, v_\xi) \mid \xi < \delta\rangle$ and $\langle \mathcal{M}_\xi \mid 0 < \xi < \delta\rangle$. Since δ is greater than 2^{κ_k} there is an unbounded set $I \subseteq \delta$ such that for $\xi \in I$, $\langle A^{u_\xi}_\ell, \dots, A^{u_\xi}_k\rangle$ is constant; say, with value $\langle B_\ell, \dots, B_k\rangle$. Unlike in [HS19], the sequence $\langle u_\xi \mid \xi \in I\rangle$ may not be $\leq^*$-decreasing because even though $u_{\xi+1} \leq^{*,k,-} v_\xi$ the latter is not stronger than u_ξ. However we can argue as follows. Let u_δ be a $\leq^{*,k}$-lower bound for the v_ξ's and strengthen it so that its first $(k-\ell)$-many measure one sets are $\langle B_\ell, \dots, B_k\rangle$. Note that for all $\vec{\nu}$ and $\xi \in I$, $u_\delta{}^\frown\vec{\nu} \leq^* v_\xi{}^\frown\vec{\nu} \leq^* u_\xi{}^\frown\vec{\nu}$. In fact, for each $\vec{\nu}$, $\langle u_\xi{}^\frown\vec{\nu} \mid \xi \in I\rangle$ is decreasing. So, $u_\delta{}^{\frown+k} \leq u_\xi{}^{\frown+k}$ for all $\xi \in I$ (recall Definition 5.4).

Let $y = y_\delta := \theta \cap (\bigcup_{\xi<\delta} \mathcal{M}_\xi)$. For each $\xi \in I \cup \{\delta\}$, let X_ξ denote the collection of $\mathbb{P}_\lambda$-names decided to be $\dot{b} \cap y_\xi$ by a $\bar{n}$-extension of u_ξ.[15] By passing from δ to the unbounded set I, re-enumerate $\langle u_\xi \mid \xi \in I\rangle$ by $\langle u_\xi \mid \xi < \delta\rangle$. Let us summarize the properties of $\langle u_\xi \mid \xi < \delta\rangle$:

(1) For all $\xi < \delta$, each $\bar{n}$-step extension of u_ξ decides the value of $\dot{b} \cap y_\xi$ and X_ξ is the collection of these values; $y_\xi = \mathcal{M}_\xi \cap \theta$ and $u_\xi \in \mathcal{M}_{\xi+1}$

(2) For all ξ the measure one sets of u_ξ at coordinates $[\ell, k]$ are constant with value $\langle B_\ell, \dots, B_k\rangle$; their a-parts at these coordinates are also constant (and equal to the ones from the original condition u).

(3) For all $\vec{\nu} \in \prod_{l\leq i\leq k} B_i$, $\langle u_\xi{}^\frown\vec{\nu} \mid \xi < \delta\rangle$ is $\leq^*$-decreasing.

Claim 5.8.1. *For each $\tau \in X_\eta$ and $\xi \leq \eta$, $\tau \cap y_\xi \in X_\xi$. In addition, there is $\xi^* < \delta$ such that if τ and σ are distinct members of X_δ then there is $\xi < \xi^*$ such that $\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda} \tau \cap y_\xi \neq \sigma \cap y_\xi$.*

Proof of claim. Let $\xi \leq \eta$, and $\tau \in X_\eta$. Let $\vec{\nu}$ be such that $u_\eta{}^\frown\vec{\nu}$ forces $\dot{b} \cap y_\eta = \tau$, and let $\sigma \in X_\xi$ be such that $u_\xi{}^\frown\vec{\nu}$ forces $\dot{b} \cap y_\xi = \sigma$. Then, since $u_\eta{}^\frown\vec{\nu} \leq^* u_\xi{}^\frown\vec{\nu}$, we have that $u_\eta{}^\frown\vec{\nu} \Vdash_{j(\mathbb{P}_\lambda)} \dot{b} \cap y_\xi = \sigma$, and so $\tau \cap y_\xi = \sigma \in X_\xi$.

The second claim follows from $|X_\delta| < \delta$. More precisely, first $|X_\delta| < \delta$ because $|X_\delta|$ is at most the number of all possible $\bar{n}$-extensions of each u_ξ, which is at most $2^\kappa_k < \delta$. Now, every two distinct elements of X_δ split at some $\xi < \delta$. When we take the supremum of all those ξ's we get $\xi^* < \delta$, since it is a supremum of less than δ- many ordinals and δ is regular. □

Roughly speaking the above claim says that no more splittings are forced past level ξ^*. The next claim plays the role of the usual branching lemma:

Claim 5.8.2. *For every ordinal $\xi^* < \xi < \delta$ and $\bar{r} \leq \pi(u_\xi)$ with $\bar{r} \in \mathcal{M}_{\xi+1}$ there is a set $z \in \mathcal{M}_{\xi+1}$ and a condition $r \in j(\mathbb{P}_\lambda)$ as follows:*

[15] Using that $u_\delta{}^{\frown+k} \leq u_\xi{}^{\frown+k}$ for all $\xi \in I$ one can show that any $\bar{n}$-extension of u_δ decides the value of $\dot{b} \cap y_\delta$. Thus, it is meaningful to consider X_δ.

(1) $y_\xi \subseteq z \subseteq y_{\xi+1}$;
(2) $r \leq u_\xi$ *and* $\pi(r) \leq \bar{r}$;
(3) $r \Vdash_{j(\mathbb{P}_\lambda)}$ "$\dot{b} \cap z = \tau$" *but* $\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda}$ "$\tau \neq \sigma \cap z$" *for all* $\sigma \in X_{\xi+1}$.

Proof of claim. Assume otherwise and let ξ and $\bar{r}$ be counter-examples. Let $u \leq u_\xi$ be such that $\pi(u) \leq \bar{r}$ and u decides a value for $\dot{b} \cap y_\xi$. Let us work in an extension of $\mathcal{M}_{\xi+1}$ by a V-generic $\bar{G} \subseteq \mathbb{P}_\lambda$ with $\pi(u) \in \bar{G}$.

Set

$$d = \{\tau_{\bar{G}} \mid \exists z \supseteq y_\xi \, \exists r \in j(\mathbb{P}_\lambda)/\bar{G} \; (r \leq u \wedge \pi(r) \leq \bar{r} \wedge r \Vdash_{j(\mathbb{P}_\lambda)} \dot{b} \cap z = \tau)\}.$$

Since $\dot{b}_G \notin V[\bar{G}]$, d cannot be a branch. So, there is $z \in \mathcal{M}_{\xi+1}$ and $r_1, r_2 \in j(\mathbb{P}_\lambda)/\bar{G}$ with $r_1, r_2 \leq u_\xi$ and $\pi(r_1), \pi(r_2) \leq \bar{r}$, and τ_1, τ_2, τ such that

- $r_i \Vdash \dot{b} \cap z = \tau_i$, for $i = 1, 2$.
- $r_i \Vdash \dot{b} \cap y_\xi = \tau$, for $i = 1, 2$.
- $(\tau_1)_{\bar{G}} \neq (\tau_2)_{\bar{G}}$.

Here we use that by definition of d, since r_1 and r_2 are below u, they force that $\dot{b} \cap y_\xi$ is the same τ that u decides.

Since we assume that (3) fails there are $\sigma_1, \sigma_2 \in X_{\xi+1}$ such that $(\tau_i)_{\bar{G}} = (\sigma_i)_{\bar{G}} \cap z$ for $i = 1, 2$. So, $(\sigma_1)_{\bar{G}} \neq (\sigma_2)_{\bar{G}}$ but $(\sigma_1)_{\bar{G}} \cap y_\xi = (\sigma_2)_{\bar{G}} \cap y_\xi = \tau$. This contradicts Claim 5.8.1 above. □

The proof of the above shows that, for each $\bar{r} \leq \pi(u_\xi)$, there is a dense set of conditions r below the *meet* of u_ξ and $\bar{r}$ which satisfy the conclusion of the claim; call it $D_{\xi,\bar{r}}$. More formally, for every $w \leq u_\xi$ such that $\pi(w) \leq \bar{r}$ there is $w' \leq w$ in $D_{\xi,\bar{r}}$. Combining this with Strong Prikry lemma, we get:

Claim 5.8.3. *For every* $\xi^* < \xi < \delta$ *and* $\bar{r} \leq^{*,k,-} \pi(u_\xi)$ *with* $\bar{r} \in \mathcal{M}_{\xi+1}$ *there is* $r \leq^{*,k,-} u_\xi$ *with* $r \in \mathcal{M}_{\xi+1}$, $\pi(r) \leq^{*,k,-} \bar{r}$ *and a set* $z \in \mathcal{M}_{\xi+1}$, *such that*

(1) $y_\xi \subseteq z \subseteq y_{\xi+1}$;
(2) *every* $\bar{n}$*-step extension of* r *forces that* $\dot{b} \cap z = \tau$ *for some* τ, *but* $\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda}$ "$\tau \neq \sigma \cap z$" *for all* $\sigma \in X_{\xi+1}$.

Proof of claim. Work in $\mathcal{M}_{\xi+1}$. Let $r \leq^{*,k,-} u_\xi$ with $\pi(r) \leq^{*,k,-} \bar{r}$ be such that for some $n \geq \bar{n}$, every n-step extension of r is in $D_{\xi,\bar{r}}$. For all n-step extension $r^\frown\vec{\nu}$, let $z_{\vec{\nu}}$ witness membership in $D_{\xi,\bar{r}}$ and let $z = \bigcup_{\vec{\nu}} z_{\vec{\nu}}$. Let $r' \leq^{*,k,-} r$ be such that every $\bar{n}$-step extension of r' decides $\dot{b} \cap z$. Then r' is as desired. □

For simplicity, say $\xi^* = 0$. Let us define a sequence $\langle r_\xi, s_\xi \mid \xi < \delta\rangle$ by induction as follows. Let $\bar{r} \leq^{*,k} \pi(u_0)$ be such that $\bar{r}^{\frown +k} \leq \pi(u_\delta)^{\frown +k}$ (see Lemma 5.6(2)). Let $r_0 \leq^{*,k,-} u_0$ be as in the above claim when regarded for $\xi = 0$ and $\bar{r}$. By further $\leq^{*,k,-}$-extending r_0 we may assume that all of its $\bar{n}$-extensions decide $\dot{b} \cap y$. Let $s_0 \leq^{*,k} \bar{r}$ be such that

$$s_0{}^\frown\vec{\nu} = \pi(r_0)^\frown\vec{\nu} \text{ for all } \vec{\nu} \in \textstyle\prod_{i=\ell}^{k} A_i^{\pi(r_0)}.$$

To ensure this choice we make use of Remark 5.7 in page 26.

Suppose that $\langle r_\zeta, s_\zeta \mid \zeta < \xi\rangle$ has been defined so that

- $\vec{s} := \langle s_\zeta \mid \zeta < \xi\rangle$ is $\leq^{*,k}$-decreasing;
- $\pi(r_\zeta)^\frown\vec{\nu} = s_\zeta{}^\frown\vec{\nu}$ for all $\vec{\nu} \in \prod_{i=\ell}^{k} A_i^{\pi(r_\zeta)}$ and $\zeta < \xi$.

Since $\xi < \delta < \kappa_{k+1}$ there is a $\leq^{*,k}$-lower bound for $\vec{s}$; call it s^*. By Lemma 5.6(2) there is $\bar{r}_\xi \leq^{*,k} \pi(u_\xi)$ such that $\bar{r}_\xi{}^{\frown+k} \leq s^{*\frown+k}$. Apply the previous claim with respect to ξ and $\bar{r}_\xi$ to produce $r_\xi \leq^{*,k,-} u_\xi$ such that $\pi(r_\xi) \leq^{*,k,-} \bar{r}_\xi$. As before, we may assume that all the $\bar{n}$-extensions of r_ξ decide $\dot{b} \cap y$. Using Remark 5.7, let $s_\xi \leq^{*,k} s^*$ as above.

The upshot of the previous construction is a sequence $\langle r_\xi \mid \xi < \delta\rangle$ such that $\langle \pi(r_\xi)^\frown\vec{\nu} \mid \xi < \delta\rangle$ is $\leq^*$-decreasing, provided $\vec{\nu}$ is a Prikry point common to the first $(k-\ell)$-many measure one sets of the r_ξ's. By passing to an unbounded subset of δ, we may assume that the measure one sets of the r_ξ's at coordinates ℓ, ..., k are in fact constant. Thus, by construction, $\langle \pi(r_\xi) \mid \xi < \delta\rangle$ is ${}^{\frown+k} \leq$-decreasing so we may take a ${}^{\frown+k} \leq$-lower bound, $\bar{u}$ (Lemma 5.6(1)). Note that $\bar{u}^{\frown+k} \leq \pi(u_\delta)^{\frown+k}$.

Claim 5.8.4. *$\langle r_\xi \mid \xi < \delta\rangle$ and $\bar{u}$ are as in the splitting lemma.*

Proof of claim. It is enough to prove Clause (4) of the lemma. Let $\xi < \zeta$ and s, s' be $\bar{n}$-extensions of r_ξ and r_ζ. Let τ and τ' be the $\mathbb{P}_\lambda$-values of $\dot{b} \cap y$ decided by s and s', respectively. By construction of r_ξ, τ is incompatible with members of $X_{\xi+1}$ as forced by $\pi(u_\xi)$ (by Clause (3) in Claim 5.8.2) Namely, for all $\sigma \in X_{\xi+1}$,

$$\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda} \tau \cap y_{\xi+1} \neq \sigma.$$

On the other hand, since $s' \leq^{\bar{n}} r_\zeta \leq^* u_\zeta$, we have that $\tau' \in X_\zeta$. Hence, by Claim 5.8.1, $\tau' \cap y_{\xi+1} \in X_{\xi+1}$. So,

$$\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda} \tau \cap y_{\xi+1} \neq (\tau' \cap y_{\xi+1}).$$

Since $\bar{u}^{\frown+k} \leq \pi(u_\xi)^{\frown+k}$ we get that $\bar{u}$ also $\mathbb{P}_\lambda$-forces "$\tau \neq \tau'$". □

The above completes the proof of the splitting lemma. □

Remark 5.9. The same argument as above works working below any $\bar{v} \leq^{*,k,-} \pi(u)$. In that case the Splitting Lemma gives a condition $\bar{u} \leq^{*,k,-} \bar{v}$.

The next theorem completes the proof of Theorem 1 in page 2.

Theorem 5.10. ITP(λ) *holds in a generic extension by $\mathbb{P}_\lambda$.*

Proof. For simplicity, suppose the initial condition $u \in j(\mathbb{P}_\lambda)$ has length 0. Fix measurable cardinals $\langle \delta_n \mid n < \omega\rangle$ such that $2^{\kappa_{\bar{n}+n}} < \delta_n < \kappa_{\bar{n}+n+1}$; say their measurability is witnessed by δ_n-complete measures $\langle \mathcal{U}_n \mid n < \omega\rangle$.

We define a tree of conditions $\langle r_{\vec{\sigma}} \mid \vec{\sigma} \in \prod_{n<m} Y_m,\ m < \omega\rangle \subseteq j(\mathbb{P}_\lambda)$, where each $Y_m \in \mathcal{U}_m$, a $\leq^*$-decreasing sequence $\langle \bar{u}_n \mid n < \omega\rangle \subseteq \mathbb{P}_\lambda$ and an $\subseteq$-increasing sequence $\langle y_n \mid n < \omega\rangle$ in $\mathcal{P}_\lambda(\theta)$ with the following properties:

(1) $r_{\vec{\sigma}} \leq^* r_{\vec{\tau}}$ in case $\vec{\tau} \sqsubseteq \vec{\sigma}$;
(2) $\bar{u}_{n+1} \leq^{*,n,-} \bar{u}_n$;
(3) For each $\vec{\sigma} \in \prod_{n<m} Y_m$ and $\xi \in Y_m$, $r_{\vec{\sigma}^\frown\langle\xi\rangle} \leq^{*,m,-} r_{\vec{\sigma}}$;
(4) For each $\vec{\sigma}$ all the $\bar{n}$-extensions of $u_{\vec{\sigma}}$ decide $\dot{b} \cap y_{|\vec{\sigma}|}$;
(5) For incompatible $\vec{\sigma}$ and $\vec{\tau}$, any two $\bar{n}$-extensions of $u_{\vec{\sigma}}$ and $u_{\vec{\tau}}$ decide incompatible values for $\dot{b} \cap y_{|\vec{\sigma}|}$ and $\dot{b} \cap y_{|\vec{\tau}|}$, as $\mathbb{P}_\lambda$-forced by $\bar{u}_{|\vec{\sigma}\cap\vec{\tau}|}$,

By induction on $|\vec{\sigma}|$. First, apply the Splitting Lemma (Lemma 5.8) to the triple $\langle u, \delta_{\bar{n}}, \pi(u)\rangle$ to obtain $\langle r_\xi \mid \xi < \delta_{\bar{n}}\rangle$ and $\bar{u}_{\bar{n}}$, so that:

- $r_\xi \leq^{*,\bar{n},-} u$ for all $\xi < \delta_0$;
- $\bar{u}_0{}^{\curvearrowright+\bar{n}} \leq \pi(r_\xi)^{\curvearrowright+\bar{n}}$ for all $\xi < \delta_0$;
- $\bar{u}_0 \leq^{*,\bar{n},-} \pi(u)$.

Since $\bar{u}_0{}^{\curvearrowright+\bar{n}} \leq \pi(r_0)^{\curvearrowright+\bar{n}}$, there is $w \leq^{*,\bar{n}} \pi(r_0)$ such that $w^{\curvearrowright+\bar{n}} \leq \bar{u}_0{}^{\curvearrowright+\bar{n}}$ (by Lemma 5.6). Apply the Splitting Lemma to $\langle r_0, \delta_1, w\rangle$ and get in return a sequence $\langle r_{0,\eta} \mid \eta < \delta_1\rangle$ and $\bar{u}_1(0)$. As before, the following hold:

- $r_{0,\eta} \leq^{*,\bar{n}+1,-} r_0$ for all $\eta < \delta_1$;
- $\bar{u}_1(0)^{\curvearrowright+\bar{n}+1} \leq \pi(r_{0,\eta})^{\curvearrowright+\bar{n}+1}$ for all $\eta < \delta_1$;
- $\bar{u}_1(0) \leq^{*,\bar{n}+1,-} w$.

We keep applying the Splitting lemma for r_ξ, $\xi < \delta_0$ by induction of ξ. Suppose we have defined $\langle\langle\langle r_{\zeta,\eta} \mid \eta < \delta_1\rangle, \bar{u}_1(\zeta)\rangle \mid \zeta < \xi\rangle$, so that $\langle \bar{u}_1(\zeta) \mid \zeta < \xi\rangle$ is $\leq^*$-decreasing except for the first $\bar{n}+1$ measure one sets; more formally, $\langle u_1(\zeta)^\frown\vec{\nu} \mid \zeta < \xi\rangle$ is $\leq^*$-decreasing provided $\vec{\nu} \in \bigcap_{\zeta<\xi}\prod_{i\leq\bar{n}} A_i^{\bar{u}_1(\zeta)}$.

► If $\xi = \zeta+1$, mimicking the argument in Lemma 5.6(2), find an auxiliary condition $z \leq^{*,\bar{n},-} \pi(r_\xi)$ with $z^{\curvearrowright+\bar{n}} = \bar{u}_1(0)^{\curvearrowright+\bar{n}}$ (i.e., $z^{\curvearrowright+\bar{n}} \leq \bar{u}_1(0)^{\curvearrowright+\bar{n}}$ and $\bar{u}_1(0)^{\curvearrowright+\bar{n}} \leq z^{\curvearrowright+\bar{n}}$). Invoke the Splitting Lemma for $\langle r_\xi, \delta_1, z\rangle$.

► ξ is limit. Again, first we need an auxiliary condition. For ease of notation, for the rest of the construction we assume that $\bar{n} = 0$. The general case is analogous.

Claim 5.10.1. *There is a condition w such that $w \leq^{*,\bar{n}} \pi(r_\xi)$ and $w^\frown\nu \leq^* \bar{u}_1(\zeta)^\frown\nu$ for all $\nu \in A_0^{\bar{u}_1(\zeta)}$ and $\zeta < \xi$.*

Proof. Let us show how to define such a condition:

<u>EBPF-part:</u> The EBPF-parts of $u_1(\zeta)$ (for $\zeta < \xi$) and $\pi(r_\xi)$ take the form

$$\langle(a_0, A_0^\zeta, f_0^\zeta)\rangle^\frown p_\zeta \restriction [1,\omega),$$
$$\langle(a_0, A_0, f_0)\rangle^\frown p_{\pi(r_\xi)} \restriction [1,\omega),$$

where $A_0^\zeta \subseteq A_0$ and $f_0 \subseteq f_0^\zeta$ (because $\bar{u}_0{}^{\curvearrowright+0} \leq \pi(r_\xi)^{\curvearrowright+0}$).

The EBPF-part of w is defined as

$$p_w := \langle(a_0, A_0, \textstyle\bigcup_{\zeta<\xi} f_0^\zeta)\rangle^\frown \textstyle\bigwedge_{\zeta<\xi}(p_\zeta \restriction [1,\omega)),$$

where the latter is just a lower bound for the displayed conditions.

Mitchell-part: Since $\langle \bar{u}_1(\zeta)^\frown\nu \mid \zeta < \xi\rangle$ is $\leq^*$-decreasing we have

$$\mathrm{dom}(c_\zeta) = \mathrm{dom}(c_\zeta{}^\frown\nu) \supseteq \mathrm{dom}(c_{\bar{\zeta}}{}^\frown\nu) = \mathrm{dom}(c_{\bar{\zeta}}) \supseteq \mathrm{dom}(c^{\pi(r_\xi)}),$$

where c_ζ is the Mitchell-part of $\bar{u}_1(\zeta)$. Let $\mathrm{dom}(c_w) := \bigcup_{\zeta<\xi}\mathrm{dom}(c_\zeta)$.

For each $\alpha \in \mathrm{dom}(c_w)$ we choose $\mathrm{dom}(c_w(\alpha))$ in the obvious way towards ensuring $(p_w, c_w) \in \mathbb{M}$. For each $\vec{\pi} \in \mathrm{dom}(c_w)$ define $c_w(\alpha, \vec{\pi})$ as:

► If $\alpha \notin \mathrm{dom}(c^{\pi(r_\xi)})$ then

$$c_w(\alpha, \vec{\pi}) := \begin{cases} \bigwedge\{c_\zeta(\alpha, \vec{\sigma}_\zeta) \mid \alpha \in \mathrm{dom}(c_\zeta), \langle \pi_0\rangle \in \mathrm{dom}(c_\zeta(\alpha))\}, & \text{if } |\vec{\pi}| \geq 2;\\ \{\langle \varnothing, \mathbb{1}_{\mathbb{P}\restriction\alpha}\rangle\}, & \text{otherwise,}\end{cases}$$

where $\vec{\sigma}_\zeta$ is the sequence of the projections of $\vec{\pi}$ under $\pi_{\mathrm{mc}(a^w\cap\alpha),\mathrm{mc}(a^{\bar{u}_1(\zeta)}\cap\alpha)}$'s.

► If $\alpha \in \mathrm{dom}(c^{\pi(r_\xi)})$ then $c_w(\alpha, \vec{\pi})$ is defined as above replacing the $\mathbb{P}\restriction\alpha$-name in the second case by $c^{\pi(r_\xi)}(\alpha, \langle\pi\rangle)$.

We leave to the reader checking that $(p_w, c_w) \in \mathbb{M}$ and $(p_w, c_w) \leq^{*,0} (p_{\pi(r_\xi)}, c^{\pi(r_\xi)})$. Also note that $(p_w, c_w)^\frown\nu \leq^* (p_\zeta, c_\zeta)^\frown\nu$ for all $\nu \in A_0^\zeta$.

Sharon part: Let us describe what to do at the first Sharon-like strategy. For each $\nu \in A_0$ set $I_\nu := \{\zeta < \xi \mid \nu \in A_0^\zeta\}$. Let w_ν denote a $\leq^*$-lower bound for $\langle \pi(r_\xi)^\frown\nu\rangle^\frown\langle \bar{u}_1(\zeta)^\frown\nu \mid \zeta \in I_\nu\rangle$ (this is possible as $\xi < \delta_0 < \kappa_1$).

Next, diagonalize the sequence $\langle w_\nu \mid \nu \in A_0\rangle$ thus finding $w \leq^{*,0} \pi(r_\xi)$ (in fact, the first coordinate of w is (p_w, c_w)) such that $w^\frown\nu \leq^* w_\nu$ for all $\nu \in A_0$. Note that for each $\zeta < \xi$ and $\nu \in A_0^\zeta$, $w^\frown\nu \leq^* \bar{u}_1(\zeta)^\frown\nu$. □

After this we get an auxiliary condition w such that $w \leq^{*,0} \pi(r_\xi)$ and $w^\frown\nu \leq^* \bar{u}_1(\zeta)^\frown\nu$ for $\nu \in A_0^\zeta$, $\zeta < \xi$. Apply the Splitting Lemma to the triple $\langle r_\xi, \delta_1, w\rangle$ and obtain $\langle r_{\xi,\eta} \mid \eta < \delta_1\rangle$ and $\bar{u}_1(\xi)$. Clearly,

$$\bar{u}_1(\xi)^\frown\nu \leq^* \bar{u}_1(\zeta)^\frown\nu \text{ for all } \nu \in \textstyle\bigcap_{\zeta<\xi} A_0^\zeta.$$

Thereby we get $\langle r_{\xi,\eta} \mid \xi < \delta_0, \eta < \delta_1\rangle$ and $\langle \bar{u}_1(\xi) \mid \xi < \delta_0\rangle$. Let us show how one stabilizes the first two measure one sets of $r_{\xi,\eta}$, $\langle A_0^{\xi,\eta}, A_1^{\xi,\eta}\rangle$. [16] This will enable us to take lower bounds upon the $\bar{u}_1(\xi)$'s. For each $\xi < \delta_0$ let $\Phi_\xi \colon \eta \mapsto \langle A_0^{\xi,\eta}, A_1^{\xi,\eta}\rangle$. Since δ_1 is measurable and $2^{\kappa_1} < \delta_1$ we find $B_{1,\xi} \in \mathcal{U}_1$ where Φ_ξ is constant; say with value $\langle A_0^\xi, A_1^\xi\rangle$. Since $\delta_0 < \kappa_1$ it follows that $B_1 := \bigcap_{\xi<\delta_0} B_{1,\xi} \in \mathcal{U}_1$ and $A_1 := \bigcap_{\xi<\delta_0} A_1^\xi$ is $E_{1,\mathrm{mc}(a_1)}$-large, where a_1 is common among the $r_{\xi,\eta}$'s (by construction). Finally, use that $2^{\kappa_0} < \delta_0$ to find $B_0 \in \mathcal{U}_0$ for which A_0^ξ is constant; say with value A_0. Let us now look at the sequences $\langle r_{\xi,\eta} \mid \xi \in B_0, \eta \in B_1\rangle$ and $\langle \bar{u}_1(\xi) \mid \xi \in B_0\rangle$. By construction,

$$\bar{u}_1(\xi)^{\frown+1} \leq \pi(r_{\xi,\eta})^{\frown+1} \text{ for all } \eta < \delta_1,$$

[16] Again, assume that $\bar{n} = 0$. In the general case we would be stabilizing measure one sets at coordinates $i \leq \bar{n} + 1$. The argument is the same.

so that the first two measure one sets of $\bar{u}_1(\xi)$ are $\pi_{\mathrm{mc}(a_0),\mathrm{mc}(a_0\cap\lambda)}$ "A_i^ξ. Shrink the second measure one set of each $r_{\xi,\eta}$ (i.e., A_1^ξ) to A_1. Do the same with $\bar{u}_1(\xi)$ using $\pi_{\mathrm{mc}(a_1),\mathrm{mc}(a_1\cap\lambda)}$ "A_1. Keep denoting the resulting conditions by $r_{\xi,\eta}$ and $\bar{u}_1(\xi)$. Note that now we are in conditions of taking a $\leq^{*,1}$-lower bound upon the $\langle \bar{u}_1(\xi) \mid \xi \in B_0\rangle$; call this $\bar{u}_1$. Finally, let $y_1 := \bigcup_{\xi\in B_0,\eta\in B_1} y_{\xi,\eta}$ where $y_{\xi,\eta} \in \mathcal{P}_\lambda(\theta)$ are given by the Splitting Lemma.

Recalling items (1)–(5) at the beginning of the proof (page 31) one confirms that $\langle r_{\xi,\eta} \mid \xi \in B_0, \eta \in B_1\rangle$ and $\bar{u}_1$ are as wished. In general one proceeds by induction as explained above, allowing the B_i's to get shrunk. Since the measures $\mathcal{U}_n$ are countably complete, we are fine in taking intersections of these sets at the end; these will be the final Y_n's supporting the tree of $r_{\vec{\sigma}}$'s.

Let us now complete the proof of ITP(λ). As of now we have defined

$$\langle r_{\vec{\sigma}} \mid \vec{\sigma} \in \textstyle\prod_{n<m} Y_n,\, n<\omega\rangle,\ \langle \bar{u}_1 \mid n<\omega\rangle \text{ and } \langle y_n \mid n<\omega\rangle,$$

in the ground model, V. Let $\bar{u}_\omega$ be a $\leq^*$-lower bound for $\langle \bar{u}_n \mid n<\omega\rangle$ and let $G \subseteq \mathbb{P}_\lambda$ a V-generic with $\bar{u}_\omega \in G$.[17]

Claim 5.10.2. ITP(λ) *holds in* $V[G]$.

Proof of claim. Since $\dot{b}$ was forced to be an ineffable branch which is $<\lambda$-approximated in $V[G]$, letting $c \in \mathcal{P}_\lambda(\theta)^{V[G]}$ with $c \supseteq \bigcup_{n<\omega} y_n$ we infer that $b \cap c \in V[G]$. Inside $V[G]$ the product $\prod_{n<\omega} \delta_n$ has cardinality λ.

For each $f \in \prod_{n<\omega} Y_n$ let r_f be a $\leq^*$-lower bound for $\langle r_{f\restriction n} \mid n<\omega\rangle$. Such a condition exists because the G-part of each $r_{f\restriction n}$ is captured by $\bar{u}_\omega$.

More precisely, we take a lower bound of an ω decreasing sequence $r_{f\restriction n}, n < \omega$ in $V[G]$, such that for some u, each of them projects to u. First, for each i, the a^i-part of these conditions is the same on a tail end, so that's what we take, the measure one part A_i projects to (by taking a subset) to A_i^u, so when constructing r_f we take the inverse image of A_i^u, to $\max(a_i)$. And we can take union of the Cohen parts using a covering argument, since by the the Prikry lemma the number of possible values is $< \kappa^+$, which matches the closure of the Cohens.

By construction, every $\bar{n}$-extension of r_f decides the value of $\dot{b} \cap c$. Let s_f be one of such extensions and let $b_f \in V[G]$ be the decided value. Since $\mathbb{P}_\lambda$ is λ-cc and $\dot{b} \cap c$ has size $<\lambda$, there are $<\lambda$-many possible b_f's. Working inside $V[G]$, consider the map $\Phi\colon f \mapsto b_f$. Using Clause (5) in page 31 one can show that Φ is one-to-one. Therefore there are at least λ-many such b_f's. This produces the desired contradiction and establishes ITP(λ) in $V[G]$. □

This completes the proof of Theorem 5.10. □

[17]Note that the arguments at the beginning of this section (see p.24) hold for an arbitrary V-generic filter and thus they remain valid for our particular G with $\bar{u}_\omega \in G$.

6. Open question

Question 1. *What configuarions can we get by by plugging other Prikry-type forcings into the Mitchell-like functor of §2? More generally, can we show an abstract lemma about combining Mitchell with Prikry?*

Question 2. *Can we generalize Theorem 1 to the case where the cardinal κ has uncountable cofinality?*

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